EFFECT OF AN ELLIPTIC DISK OF ORTHOTROPIC MATERIAL ON THE STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE

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CHAPTER I

INTRODUCTION

If a thin flat plate is cut from an orthotropic material (i.e., material possessing three mutually perpendicular planes of elastic symmetry) such that the plane of the plate is parallel to a plane of elastic symmetry, then the plate is said to be orthotropic and it will contain two perpendicular axes of symmetry. It is the purpose of this dissertation to consider the state of stress in a large rectangular plate of this type having a small elliptic disk of a different orthotropic material at its center. Both the axes of symmetry of the plate and the axes of elastic symmetry of the disk are taken parallel to the edges of the plate. The dimensions of the disk are considered extremely small relative to those of the plate. As shown in figure 1, the center of the disk will be taken as the origin of the coordinate system with the major and minor axes of the elliptic disk coincident with the coordinate axes. The plate will be subjected to a uniform tension S along two opposite edges and the resulting stresses in the plate and in the disk will be determined. Throughout this discussion it is of course assumed that strains are small and will remain within the limits of perfect elasticity.

A second objective of this dissertation is to show that expressions for stresses of several special problems in elasticity can be obtained as limiting cases of this more general problem. Some of these have already been solved. Professor C.B. Smith[1] treated the problem of an infinite

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[1] C. B. Smith, Effect of Elliptic or Circular Holes on the Stress Distribution in Plates of Wood or Plywood Considered as Orthotropic Materials, U. S. Forest Products Laboratory Report No. 1510, 1944.

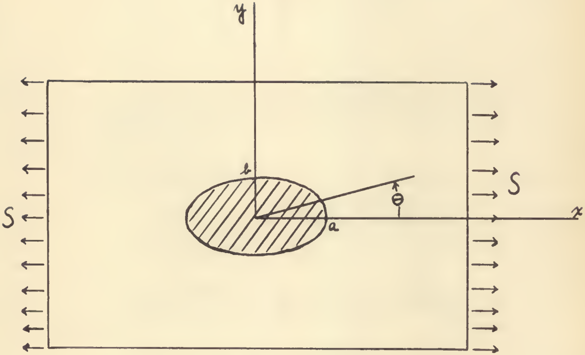

Figure 1. — Elliptic Disk of Orthotropic Material in an Orthotropic Plate

rectangular orthotropic plate with a small elliptic or circular hole at its center, and Dr. A. J. Owens[2] treated the problem of an infinite orthotropic plate with a small elliptic hole at the center filled with a rigid elliptic disk. Dr. J. C. English[3] treated the problem of an infinite rectangular orthotropic plate containing a circular hole filled by an isotropic material and then showed that the problems of Smith and Owens were obtained as special cases. Once the stresses for the more general problem treated here are obtained, no great difficulty is imposed in deriving a few special cases although, in some instances, the forms in which the stresses appear may not coincide with those previously derived by other authors. As an example, one particular problem known to have been solved is that of the elliptic isotropic disk in an infinite isotropic plate. L. H. Donnell[4] determined the stress distribution for this case but his use of a different coordinate system and method of attack would make a direct comparison with the results of this dissertation decidedly unprofitable at the present time.

In the chapter which follows, general expressions for the stresses in the orthotropic plate and in the orthotropic disk are derived by the use of complex functions. The procedure entails a discussion of the general equations of plane stress for orthotropic materials, application of mapping functions, and finally, derivation of the stresses from series which are subjected to suitable restrictions and boundary conditions.

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[2] A. J. Owens and C. B. Smith, "Effect of a Rigid Elliptic Disk on the Stress Distribution in an Orthotropic Plate," Quarterly of Applied Mathematics, Vol. IX, 1951, pp. 329-333.

[3] Julius Clyde English, "Stress Concentrations in Infinite Orthotropic Plates" (unpublished Ph.D. dissertation, University of Florida, 1952).

[4] L. H. Donnell, "Stress Concentrations Due to Elliptical Discontinuities in Plates Under Edge Forces," Theodore von Karman Anniversary Volume, Pasadena, California: California Institute of Technology, 1941, pp. 293-309.

Chapter III is reserved for a treatment of special problems whose stresses can be obtained by specializing the solution obtained for the problem considered.

CHAPTER II

SOLUTION OF THE PROBLEM

General Equations of Plane Stress

The solution of this elasticity problem involves finding general expressions for the stresses such that the state of stress at every point in the object under consideration may be specified. These stresses are obtained as derivatives of general stress functions.

Since the stress components vary over the area of the plate (and disk), when we arrive at the boundary of the elliptical disk the stress components in the plate must be in equilibrium with those in the disk so that the forces in the plate may be regarded as a continuation of those in the disk.

The mathematical formulation of the condition for compatability of stress distribution with the existence of continuous functions u and v defining the deformation results in a differential relation called the condition of compatability. If the conditions of compatability are satisfied then the existence of displacements corresponding to a given set of strain components is guaranteed.

From the above discussion it follows that three sets of conditions must be satisfied, therefore, if the function is to give us the true stresses, these conditions being:

1. the equations of equilibrium
2. the conditions of compatability
3. the boundary conditions .

Now the stress at any point in a given material is completely determined by the six components of stress at that point; i.e., three normal stresses and three shearing stresses. Similarly, the strain at any point is completely determined by the six components of strain; i.e., three extension strains and three shearing strains.[5] For the components of stress and strain, Timoshenko's[6] notation will be used. The stress components are:

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{yz}, \quad \tau_{zx}$$

If u, v and w are the components of displacement of a point in a body at a position whose coordinates are x, y and z, the components of strain are defined as follows:

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned} \tag{1}$$

Assuming that the coordinate planes coincide with the planes of elastic symmetry, the components of stress and strain for orthotropic materials are related by the following equations:[7]

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[5] The reader will find a summarization of most of the symbols and their meanings as used throughout this paper in the Appendix.

[6] S. Timoshenko and J. N. Goodier, Theory of Elasticity (2d ed.), New York: McGraw Hill Company, 1951.

[7] H. W. March, Stress-Strain Relations in Wood and Plywood Considered as Orthotropic Materials, Madison, Wisconsin: United States Forest Products Laboratory Report No. R1503, 1944, pp. 2-3.

$$\begin{aligned}
\epsilon_x &= \frac{1}{E_x}\sigma_x - \frac{\nu_{yx}}{E_y}\sigma_y - \frac{\nu_{zx}}{E_z}\sigma_z \\
\epsilon_y &= \frac{-\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y - \frac{\nu_{zy}}{E_z}\sigma_z \\
\epsilon_z &= \frac{-\nu_{xz}}{E_x}\sigma_x - \frac{\nu_{yz}}{E_y}\sigma_y + \frac{1}{E_z}\sigma_z
\end{aligned} \tag{2}$$

$$\gamma_{xy} = \frac{1}{\mu_{xy}}\tau_{xy} \ , \quad \gamma_{yz} = \frac{1}{\mu_{yz}}\tau_{yz} \ , \quad \gamma_{zx} = \frac{1}{\mu_{zx}}\tau_{zx}$$

where E_x , E_y , E_z are Young's moduli in the x- , y- and z-directions and ν is Poisson's ratio. In particular, ν_{xy} is the ratio of contraction parallel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-axis. μ is the modulus of rigidity (modulus of elasticity in shear); more specifically, μ_{xy} is the modulus of rigidity associated with the x- , y-directions.

For problems of plane stress, equations (2) reduce to:

$$\begin{aligned}
\epsilon_x &= \frac{1}{E_x}\sigma_x - \frac{\nu_{xy}}{E_y}\sigma_y \\
\epsilon_y &= \frac{-\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y \\
\gamma_{xy} &= \frac{1}{\mu_{xy}}\tau_{xy}
\end{aligned} \tag{3}$$

i.e., Hooke's Law becomes

$$\begin{aligned} \epsilon_x &= s_{11}\,\sigma_x + s_{12}\,\sigma_y \\ \epsilon_y &= s_{12}\,\sigma_x + s_{22}\,\sigma_y \\ \gamma_{xy} &= s_{33}\,\tau_{xy} \end{aligned} \tag{4}$$

where

$$s_{11} = \frac{1}{E_x}\,, \qquad s_{22} = \frac{1}{E_y}\,,$$
$$s_{12} = \frac{-\nu_{xy}}{E_x} = \frac{-\nu_{yx}}{E_y}\,, \qquad s_{33} = \frac{1}{\mu_{xy}} \tag{5}$$

Problems involving orthotropic materials can be solved by using a stress function which involves two complex functions each of which is a function of a single complex variable. Professor Smith[8] has shown that the stress distribution in an orthotropic plate in a state of plane stress can be obtained from a stress function which is the real part of

$$\Phi = F_1(x + i\alpha\epsilon y) + F_2(x + i\beta\epsilon y) \tag{6}$$

where

$$\begin{aligned} \epsilon &= (s_{22}/s_{11})^{\frac{1}{4}} = (E_x/E_y)^{\frac{1}{4}} \\ K &= \left[\tfrac{1}{2}(s_{11}s_{12})^{-\frac{1}{2}}(2s_{12} + s_{33})\right] \\ \alpha &= \left[K + (K^2 - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}} \\ \beta &= \left[K - (K^2 - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}} \end{aligned} \tag{7}$$

It will be convenient to make a slight change in the form of equation (6) and take the stress function as the real part of

$$\Phi = F_1(x + i\alpha_1 y) + F_2(x + i\alpha_2 y) \tag{8}$$

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[8] C. B. Smith, "Effect of Hyperbolic Notches on the Stress Distribution in a Wood Plate," Quarterly of Applied Mathematics, Vol. VI, 1949, pp. 452-456.

or as the real part of

$$\Phi = F_1(z_1) + F_2(z_2)$$

where

$$z_1 = x + i\alpha_1 y \quad , \quad z_2 = x + i\alpha_2 y$$
$$\alpha_1 = \alpha\epsilon \quad , \quad \alpha_2 = \beta\epsilon \tag{9}$$

The quantities α_1 and α_2 are real quantities that depend upon the elastic constants of the plate.

The stress components are then given by the real parts of the expressions

$$\begin{aligned}
\sigma_x &= -\alpha_1^2 \frac{\partial^2}{\partial z_1^2} F_1(x + i\alpha_1 y) - \alpha_2^2 \frac{\partial^2}{\partial z_2^2} F_2(x + i\alpha_2 y) \\
\sigma_y &= \frac{\partial^2}{\partial z_1^2} F_1(x + i\alpha_1 y) + \frac{\partial^2}{\partial z_2^2} F_2(x + i\alpha_2 y) \\
\tau_{xy} &= -i\alpha_1 \frac{\partial^2}{\partial z_1^2} F_1(x + i\alpha_1 y) - i\alpha_2 \frac{\partial^2}{\partial z_2^2} F_2(x + i\alpha_2 y) \quad .
\end{aligned} \tag{10}$$

In accordance with this system the following notation will be used for the plate (i.e., exterior to the disk):

$$\begin{aligned}
\Phi &= R\left\{F_1(x + i\alpha_1 y) + F_2(x + i\alpha_2 y)\right\} \\
&= R\left\{F_1(z_1) + F_2(z_2)\right\}
\end{aligned} \tag{11}$$

where $R\{\ \}$ stands for the real part of the expression in the bracket and

$$\begin{aligned}
&\alpha_1 = \alpha\epsilon \quad , \quad \alpha_2 = \beta\epsilon \quad , \quad \epsilon = (s_{22}/s_{11})^{\frac{1}{4}} \\
&K = \left[\tfrac{1}{2}(s_{11}s_{22})^{-\frac{1}{2}}(2s_{12} + s_{33})\right] \\
&\alpha = \left[K + (K^2 - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}} \quad , \quad \beta = \left[K - (K^2 - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}}
\end{aligned} \tag{12}$$

Hooke's Law being

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = s_{11}\sigma_x + s_{12}\sigma_y \\ \epsilon_y &= \frac{\partial v}{\partial y} = s_{12}\sigma_x + s_{22}\sigma_y \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = s_{33}\tau_{xy} \end{aligned} \tag{13}$$

where the elastic constants of the plate only are used.

Similarly, for the orthotropic disk within the plate the following notation will be used:

$$\begin{aligned} \overset{\circ}{\Phi} &= R\left\{\overset{\circ}{F}_1(x + i\overset{\circ}{\alpha}_1 y) + \overset{\circ}{F}_2(x + i\overset{\circ}{\alpha}_2 y)\right\} \\ &= R\left\{\overset{\circ}{F}_1(\overset{\circ}{z}_1) + \overset{\circ}{F}_2(\overset{\circ}{z}_2)\right\} \end{aligned} \tag{14}$$

where

$$\begin{aligned} &\overset{\circ}{\alpha}_1 = \overset{\circ}{\alpha}\,\overset{\circ}{\epsilon}\ , \quad \overset{\circ}{\alpha}_2 = \overset{\circ}{\beta}\,\overset{\circ}{\epsilon}\ , \quad \overset{\circ}{\epsilon} = (\overset{\circ}{s}_{22}/\overset{\circ}{s}_{11})^{\frac{1}{4}} \\ &\overset{\circ}{k} = \left[\tfrac{1}{2}(\overset{\circ}{s}_{11}\overset{\circ}{s}_{22})^{-\frac{1}{2}}(2\overset{\circ}{s}_{12} + \overset{\circ}{s}_{33})\right] \\ &\overset{\circ}{\alpha} = \left[\overset{\circ}{k} + (\overset{\circ}{k}^2 - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}}\ , \quad \overset{\circ}{\beta} = \left[\overset{\circ}{k} - (\overset{\circ}{k}^2 - 1)^{\frac{1}{2}}\right]^{\frac{1}{2}} \end{aligned} \tag{15}$$

Hooke's Law being

$$\begin{aligned} \overset{\circ}{\epsilon}_x &= \frac{\partial \overset{\circ}{u}}{\partial x} = \overset{\circ}{s}_{11}\overset{\circ}{\sigma}_x + \overset{\circ}{s}_{12}\overset{\circ}{\sigma}_y \\ \overset{\circ}{\epsilon}_y &= \frac{\partial \overset{\circ}{v}}{\partial y} = \overset{\circ}{s}_{12}\overset{\circ}{\sigma}_x + \overset{\circ}{s}_{22}\overset{\circ}{\sigma}_y \\ \overset{\circ}{\gamma}_{xy} &= \frac{\partial \overset{\circ}{v}}{\partial x} + \frac{\partial \overset{\circ}{u}}{\partial y} = \overset{\circ}{s}_{33}\overset{\circ}{\tau}_{xy} \end{aligned} \tag{16}$$

where the elastic constants used here refer only to the orthotropic material of the disk.

In deriving (6) the equations of equilibrium and the compatability equation have been satisfied[9] so that the remainder of this analysis will be devoted largely to the boundary conditions.

For the problem under consideration stresses derivable from a stress function of the form (11) which satisfy the boundary conditions at infinity and on the boundary of the disk will be sought; additionally, stresses derivable from a stress function of the form (14) which satisfy the boundary conditions along the periphery of the disk are to be determined.

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[9] Ibid., p. 453.

Boundary Conditions

The equations of the ellipse in figure 1 are given by

$$\begin{aligned} x &= a \cos\theta \\ y &= b \sin\theta \quad . \end{aligned} \tag{17}$$

If the positive direction of the normal ν be oriented as shown in figure 2, then $X_\nu ds$ and $Y_\nu ds$ represent the x- and y-components of the force per unit thickness of the plate acting on the element of arc ds of the boundary from the positive side indicated by the direction of positive ν.

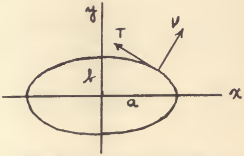


Figure 2. — Orientation of Positive Normal and Tangent on the Boundary of the Ellipse.

Then from considerations of equilibrium

$$\begin{aligned} X_\nu &= \sigma_x \cos(x,\nu) + \tau_{xy} \cos(y,\nu) \\ Y_\nu &= \tau_{xy} \cos(x,\nu) + \sigma_y \cos(y,\nu) \end{aligned} \tag{18}$$

where

$$\cos(x,\nu) = \frac{dy}{ds} \quad , \quad \cos(y,\nu) = \frac{-dx}{ds} \tag{19}$$

and

$$dx = -a \sin\theta \, d\theta \quad , \quad dy = b \cos\theta \, d\theta$$

$$ds = \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta} \quad d\theta \quad .$$

One observes that if Φ represents the stress function for the plate, then

$$X_\nu = \frac{\partial^2 \Phi}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 \Phi}{\partial x \partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial \Phi}{\partial y}\right)$$

$$Y_\nu = \frac{-\partial^2 \Phi}{\partial x \partial y}\frac{dy}{ds} - \frac{\partial^2 \Phi}{\partial x^2}\frac{dx}{ds} = \frac{-d}{ds}\left(\frac{\partial \Phi}{\partial x}\right). \qquad (20)$$

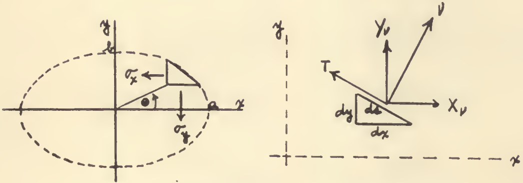


Figure 3. — Resolution of Stress Components.

It is clear that the displacements of the disk and plate must be equal at the boundary of the disk, and it will be shown later that x- and y-components of the force on the boundary per unit thickness must also be equal for the plate and disk. Consequently, along the periphery of the disk it is required that

$$u = \overset{\circ}{u}, \qquad v = \overset{\circ}{v}, \qquad X_\nu = \overset{\circ}{X}_\nu, \qquad Y_\nu = \overset{\circ}{Y}_\nu \qquad (21)$$

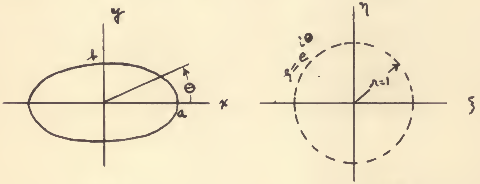
and for the plate (exterior to the disk) the boundary conditions are specified to be

$$\sigma_x\,(x = \pm\infty) = S, \quad \sigma_y\,(y = \pm\infty) = 0, \quad \tau_{xy}\begin{pmatrix} x = \pm\infty \\ y = \pm\infty \end{pmatrix} = 0 \qquad (22)$$

where S is the uniform tension applied along two opposite edges of the plate.

Preliminary Discussion on Mapping Procedure

Mapping functions which are analytic functions of complex variables will be used to solve the problem of the orthotropic plate. In order to show the general procedure involved, a brief digression will be made from the original problem in considering a simple mapping of an ellipse in the z-plane to a unit circle in the ζ -plane.



z-plane

ζ -plane

Figure 4. — Ellipse Mapped into a Unit Circle.

As is customary in complex analysis, set $z = x + iy$ where

$$x = a \cos\theta \quad , \quad y = b \sin\theta \; .$$

On the ζ-plane this ellipse is mapped into the unit circle

$$\zeta = r e^{i\theta} \quad \text{where} \quad r = 1 \; ;$$

i.e.,

$$\zeta = e^{i\theta} = \xi + i\eta$$

Hence

$$x = a\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

$$y = b\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) = \frac{b}{2i}\left(\zeta - \frac{1}{\zeta}\right)$$

so that

$$\begin{aligned} z &= x + iy \\ &= \frac{a}{2}\left(\varsigma + \frac{1}{\varsigma}\right) + \frac{ib}{2i}\left(\varsigma - \frac{1}{\varsigma}\right) \\ &= \frac{a+b}{2}\,\varsigma + \frac{a-b}{2}\,\frac{1}{\varsigma} \end{aligned}$$

Solving for ς in terms of z it follows that

$$\varsigma = \frac{z \pm \sqrt{z^2 - (a^2 - b^2)}}{a+b}$$

which is a double-valued function of z and hence contains two branches. The two branch points are found by setting the radical equal to zero, which gives

$$z = \pm\sqrt{a^2 - b^2} \quad .$$

These points are readily seen to be the two foci of the ellipse, and it is possible to take the line on the x-axis joining these two points as the branch cut.

On the x-axis, $y = 0$, so that

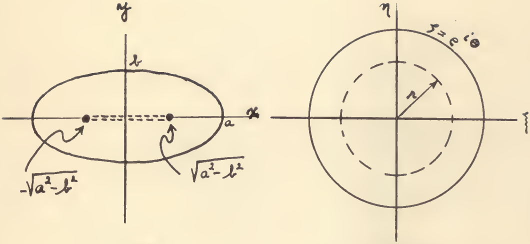
$$\varsigma = \frac{x + i\sqrt{(a^2 - b^2) - x^2}}{a+b} = \xi + i\eta \quad .$$

Hence

$$\xi = \frac{x}{a+b} \quad , \quad \eta = \frac{\sqrt{(a^2 - b^2) - x^2}}{a+b}$$

and

$$\xi^2 + \eta^2 = \frac{a-b}{a+b} \quad .$$



z-plane

ζ -plane

Figure 5. — Branch Cut of Ellipse Mapped into a Circle.

As a point moves along the x-axis between the branch points $\pm\sqrt{a^2-b^2}$ of the ellipse in the z-plane, a corresponding point moves around a circle with center at origin inside the unit circle in the ζ -plane; the radius of this circle being

$$r = \sqrt{\frac{a-b}{a+b}} \quad .$$

To insure that ζ is a single-valued function of z (i.e., to map ζ into the region outside of this circle) it will be necessary to have

$$\zeta = \left| \frac{z + \sqrt{z^2 - (a^2 - b^2)}}{a+b} \right| > \left| \sqrt{\frac{a-b}{a+b}} \right| \quad .$$

It should also be noted that as a point moves around the ellipse in the z-plane, a corresponding point moves around a unit circle in the ζ -plane.

Mapping Applied to the Orthotropic Disk

By a procedure similar to that in the preceding section, a mapping of the boundary of the orthotropic disk is effected.

In the $\mathring{z}_1$-plane let

$$\begin{aligned} \mathring{z}_1 &= x + i\mathring{\alpha}\mathring{\varepsilon} y \\ &= x + i\mathring{\alpha}_1 y \\ &= \mathring{x}_1 + i\mathring{y}_1 \end{aligned} \tag{23}$$

where $\mathring{x}_1 = a\cos\theta$, $\mathring{y}_1 = \mathring{\alpha}_1 b\sin\theta$ is the boundary of the disk. To map this curve into a unit circle on the $\mathring{\zeta}_1$-plane let

$$\mathring{\zeta}_1 = e^{i\theta} = \xi + i\eta \ .$$

Then

$$\mathring{x}_1 = \frac{a}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{a}{2}\left(\mathring{\zeta}_1 + \frac{1}{\mathring{\zeta}_1}\right)$$

$$\mathring{y}_1 = \frac{\mathring{\alpha}_1 b}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{\mathring{\alpha}_1 b}{2i}\left(\mathring{\zeta}_1 - \frac{1}{\mathring{\zeta}_1}\right)$$

and

$$\begin{aligned} \mathring{z}_1 &= \frac{a}{2}\left(\mathring{\zeta}_1 + \frac{1}{\mathring{\zeta}_1}\right) + \frac{\mathring{\alpha}_1 b}{2}\left(\mathring{\zeta}_1 - \frac{1}{\mathring{\zeta}_1}\right) \\ &= \frac{a + \mathring{\alpha}_1 b}{2}\,\mathring{\zeta}_1 + \frac{a - \mathring{\alpha}_1 b}{2}\,\frac{1}{\mathring{\zeta}_1} \end{aligned}$$

or

$$2\mathring{z}_1 = \mathring{\lambda}_1 \mathring{\zeta}_1 + \frac{\mathring{\mu}_1}{\mathring{\zeta}_1} \tag{24}$$

where

$$\begin{aligned} \mathring{\lambda}_1 &= a + \mathring{\alpha}_1 b \\ \mathring{\mu}_1 &= a - \mathring{\alpha}_1 b \end{aligned} \tag{25}$$

Hence

$$\mathring{\lambda}_1 \mathring{\zeta}_1^2 - 2\mathring{z}_1 \mathring{\zeta}_1 + \mathring{\mu}_1 = 0$$

and

$$\mathring{\zeta}_1 = \frac{\mathring{z}_1 + \sqrt{\mathring{z}_1^2 - \mathring{\lambda}_1 \mathring{\mu}_1}}{\mathring{\lambda}_1}$$

or

$$\mathring{\zeta}_1 = \frac{\mathring{z}_1 + \sqrt{\mathring{z}_1^2 - (a^2 - \mathring{\alpha}_1^2 b^2)}}{a + \mathring{\alpha}_1 b} .$$

The function $\mathring{\zeta}_1$ has branch points $\mathring{z}_1 = \pm\sqrt{a^2 - \mathring{\alpha}_1^2 b^2}$. Hence the branch cut will lie on the $\mathring{x}_1$-axis if $\mathring{\alpha}_1 b < a$, and on the $\mathring{y}_1$-axis if $\mathring{\alpha}_1 b > a$. If $\mathring{\alpha}_1 b = a$, the mapping problem is trivial.

Consider the case $\mathring{\alpha}_1 b < a$. It is seen that $\mathring{y}_1 = 0$ on the branch cut so that

$$\mathring{\zeta}_1 = \frac{\mathring{x}_1 + i\sqrt{(a^2 - \mathring{\alpha}_1^2 b^2) - \mathring{x}_1^2}}{a + \mathring{\alpha}_1 b} = \xi_1 + i\eta_1$$

from which

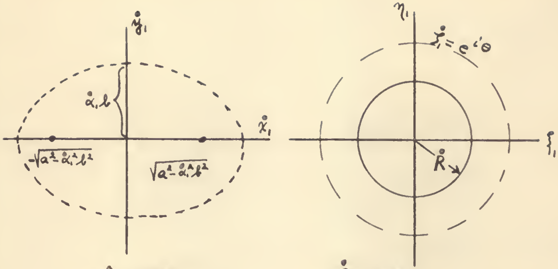
$$\xi_1 = \frac{\mathring{x}_1}{a + \mathring{\alpha}_1 b}, \qquad \eta_1 = \frac{\sqrt{(a^2 - \mathring{\alpha}_1^2 b^2) - \mathring{x}_1^2}}{a + \mathring{\alpha}_1 b}$$

and

$$\xi_1^2 + \eta_1^2 = \frac{a - \mathring{\alpha}_1 b}{a + \mathring{\alpha}_1 b} = \mathring{R}^2 .$$

Hence the branch cut in the $\mathring{z}_1$-plane maps into a circle whose radius is less than that of the unit circle in the $\mathring{\zeta}_1$-plane. To map $\mathring{\zeta}_1$ into the region outside of this circle it will always be necessary to have

$$|\mathring{\zeta}_1| = \left| \frac{\mathring{z}_1 + \sqrt{\mathring{z}_1^2 + \mathring{\alpha}_1^2 b^2 - a^2}}{a + \mathring{\alpha}_1 b} \right| > \sqrt{\frac{a - \mathring{\alpha}_1 b}{a + \mathring{\alpha}_1 b}} .$$


$\mathring{y}_1$
$\mathring{\alpha}_1 b$
$\mathring{x}_1$
$-\sqrt{a^2-\mathring{\alpha}_1^2 b^2}$
$\sqrt{a^2-\mathring{\alpha}_1^2 b^2}$
η_1
$\mathring{\zeta}_1 = e^{i\theta}$
ξ_1
$\mathring{R}$

$\mathring{z}_1$-plane $\mathring{\zeta}_1$-plane

Figure 6. — Mapping of Orthotropic Disk into an Annular Region

For the case $\mathring{\alpha}_1 b > a$, the branch cut will lie on the $\mathring{y}_1$-axis where $\mathring{x}_1 = 0$. By following the same procedure it is readily seen that the ellipse $\mathring{z}_1 = \mathring{x}_1 + i\mathring{y}_1$ maps into the unit circle $\mathring{\zeta}_1 = e^{i\theta}$ and the branch cut in the $\mathring{z}_1$-plane maps into a circle in the $\mathring{\zeta}_1$-plane with radius

$$\mathring{R} = \sqrt{\frac{\mathring{\alpha}_1 b - a}{\mathring{\alpha}_1 b + a}} \ .$$

Then to ensure mapping $\mathring{\zeta}_1$ into the outer part of this circle it will be required that

$$|\mathring{\zeta}_1| > \sqrt{\frac{\mathring{\alpha}_1 b - a}{\mathring{\alpha}_1 b + a}} \ .$$

Similar results will be obtained by carrying through the same arguments for the $\mathring{z}_2$-plane in which

$$\begin{aligned} \mathring{z}_2 &= x + i\mathring{\beta}\mathring{\epsilon} y \\ &= x + i\mathring{\alpha}_2 y \\ &= \mathring{x}_2 + i\mathring{y}_2 \end{aligned} \tag{26}$$

where $\mathring{x}_2 = a\cos\theta$, $\mathring{y}_2 = \mathring{\alpha}_2 b \sin\theta$.

Then

$$\mathring{z}_2 = \frac{a+\mathring{\alpha}_2 b}{2}\,\mathring{\zeta}_2 + \frac{a-\mathring{\alpha}_2 b}{2}\,\frac{1}{\mathring{\zeta}_2}$$

or

$$2\mathring{z}_2 = \mathring{\lambda}_2\,\mathring{\zeta}_2 + \frac{\mathring{\mu}_2}{\mathring{\zeta}_2} \tag{27}$$

where

$$\mathring{\lambda}_2 = a+\mathring{\alpha}_2 b \quad , \qquad \mathring{\mu}_2 = a-\mathring{\alpha}_2 b \tag{28}$$

The ellipse in the $\mathring{z}_2$-plane then transforms into a unit circle in the $\mathring{\zeta}_2$-plane with the final results being

$$\mathring{\zeta}_2 = \frac{\mathring{z}_2 + \sqrt{\mathring{z}_2^2 - \mathring{\lambda}_2\mathring{\mu}_2}}{\mathring{\lambda}_2}$$

where

$$|\mathring{\zeta}_2| > \sqrt{\frac{a-\mathring{\alpha}_2 b}{a+\mathring{\alpha}_2 b}} \qquad \text{or} \qquad |\mathring{\zeta}_2| > \sqrt{\frac{\mathring{\alpha}_2 b-a}{\mathring{\alpha}_2 b+a}}$$

depending on whether $\mathring{\alpha}_2 b < a$ or $\mathring{\alpha}_2 b > a$.

Hence to ensure suitable mappings of the regions in question it will be necessary to use

$$\mathring{\zeta}_1 = \frac{\mathring{z}_1 + \sqrt{\mathring{z}_1^2 - \mathring{\lambda}_1\mathring{\mu}_1}}{\mathring{\lambda}_1}$$

and

$$\mathring{\zeta}_2 = \frac{\mathring{z}_2 + \sqrt{\mathring{z}_2^2 - \mathring{\lambda}_2\mathring{\mu}_2}}{\mathring{\lambda}_2} \tag{29}$$

where it shall be required that the relations

$$|\mathring{\zeta}_1| > \sqrt{\frac{a-\mathring{\alpha}_1 b}{a+\mathring{\alpha}_1 b}} \quad , \quad |\mathring{\zeta}_2| > \sqrt{\frac{a-\mathring{\alpha}_2 b}{a+\mathring{\alpha}_2 b}} \tag{30}$$

always hold.

Continuity Conditions

In solving two-dimensional problems for orthotropic materials, the stress function (6) which is a function of two complex functions is obtained by the aid of two mapping functions; for the elliptic disk these mapping functions are $\mathring{\zeta}_1$ and $\mathring{\zeta}_2$.

It has been shown that the boundary of the ellipse in each of the $\mathring{z}_1$- and $\mathring{z}_2$-planes can be mapped into unit circles in the $\mathring{\zeta}_1$- and $\mathring{\zeta}_2$-planes respectively; and furthermore the branch cuts in the $\mathring{z}_1$- and $\mathring{z}_2$-planes can be mapped into circles with radii less than the unit circle on the $\mathring{\zeta}_1$- and $\mathring{\zeta}_2$-planes. The elliptical regions in the $\mathring{z}_1$- and the $\mathring{z}_2$-planes are then mapped into annular regions on the $\mathring{\zeta}_1$- and $\mathring{\zeta}_2$-planes.

The ring-shaped regions into which the ellipses and their branch cuts are mapped by the functions $\mathring{\zeta}_1$ and $\mathring{\zeta}_2$ suggest the use of Laurent series in $\mathring{\zeta}_1$ and $\mathring{\zeta}_2$ for this purpose. Accordingly, it will be convenient to take

$$\frac{\partial \mathring{\Phi}}{\partial x} = R\left\{ \sum_{-\infty}^{\infty} \mathring{A}_n \mathring{\zeta}_1^{\,n} + \sum_{-\infty}^{\infty} \mathring{B}_n \mathring{\zeta}_2^{\,n} \right\} \qquad (31)$$

where $\mathring{A}_n$ and $\mathring{B}_n$ are complex coefficients to be determined by the boundary conditions.

Now by virtue of equations (10) and (31) the stress components for the disk can be written as

$$\mathring{\sigma}_x = 2R\left\{\frac{-\mathring{\alpha}_1^2}{\mathring{\lambda}_1\mathring{\zeta}_1^2-\mathring{\mu}_1}\left[\sum_{-\infty}^{-1} n\mathring{A}_n\mathring{\zeta}_1^{(n+1)}+\sum_{1}^{\infty} n\mathring{A}_n\mathring{\zeta}_1^{(n+1)}\right]\right.$$

$$\left.-\frac{\mathring{\alpha}_2^2}{\mathring{\lambda}_2\mathring{\zeta}_2^2-\mathring{\mu}_2}\left[\sum_{-\infty}^{-1} n\mathring{B}_n\mathring{\zeta}_2^{(n+1)}+\sum_{1}^{\infty} n\mathring{B}_n\mathring{\zeta}_2^{(n+1)}\right]\right\}$$

$$\mathring{\sigma}_y = 2R\left\{\frac{1}{\mathring{\lambda}_1\mathring{\zeta}_1^2-\mathring{\mu}_1}\left[\sum_{-\infty}^{-1} n\mathring{A}_n\mathring{\zeta}_1^{(n+1)}+\sum_{1}^{\infty} n\mathring{A}_n\mathring{\zeta}_1^{(n+1)}\right]\right.$$

$$\left.+\frac{1}{\mathring{\lambda}_2\mathring{\zeta}_2^2-\mathring{\mu}_2}\left[\sum_{-\infty}^{-1} n\mathring{B}_n\mathring{\zeta}_2^{(n+1)}+\sum_{1}^{\infty} n\mathring{B}_n\mathring{\zeta}_2^{(n+1)}\right]\right\} \tag{32}$$

$$\mathring{\tau}_{xy} = R\left\{\frac{-i\mathring{\alpha}_1}{\mathring{\lambda}_1\mathring{\zeta}_1^2-\mathring{\mu}_1}\left[\sum_{-\infty}^{-1} n\mathring{A}_n\mathring{\zeta}_1^{(n+1)}+\sum_{1}^{\infty} n\mathring{A}_n\mathring{\zeta}_1^{(n+1)}\right]\right.$$

$$\left.-\frac{i\mathring{\alpha}_2}{\mathring{\lambda}_2\mathring{\zeta}_2^2-\mathring{\mu}_2}\left[\sum_{-\infty}^{-1} n\mathring{B}_n\mathring{\zeta}_2^{(n+1)}+\sum_{1}^{\infty} n\mathring{B}_n\mathring{\zeta}_2^{(n+1)}\right]\right\}.$$

To ensure continuity of the stress function, a relationship between the coefficients $\mathring{A}_n$ and $\mathring{A}_{-n}$, and $\mathring{B}_n$ and $\mathring{B}_{-n}$ must be obtained.

If it is assumed that $\mathring{\alpha}_1 b < a$, the branch cut will lie along the x_1-axis, and in the $\mathring{\zeta}_1$-plane we will have

$$\xi = \frac{x_1}{a+\mathring{\alpha}_1 b} \qquad \text{and} \qquad \eta = \frac{\sqrt{(a^2-\mathring{\alpha}_1^2 b^2) - x_1^2}}{a+\mathring{\alpha}_1 b}$$

so that

$$\xi^2+\eta^2 = \frac{a-\mathring{\alpha}_1 b}{a+\mathring{\alpha}_1 b} \quad .$$

As z_1 approaches the branch cut from above,

$$\dot{\zeta}_1 = \dot{R}_1 e^{i\theta} \qquad \text{where} \quad \dot{R}_1 = \sqrt{\frac{a - \mathring{\alpha}_1 b}{a + \mathring{\alpha}_1 b}} = \sqrt{\frac{\mathring{\mu}_1}{\mathring{\lambda}_1}}$$

and as z_1 approaches the branch cut from below

$$\mathring{\zeta}_1 = \mathring{R}_1 e^{-i\theta} .$$

If $\frac{\partial \dot{\Phi}}{\partial x}$ is continuous, then it is required that

$$\sum_{-\infty}^{\infty} \mathring{A}_n \dot{R}_1^{n} e^{in\theta} = \sum_{-\infty}^{\infty} \mathring{A}_n \mathring{R}_1^{n} e^{-in\theta} .$$

Hence

$$\mathring{A}_{-n} \mathring{R}_1^{-n} = \mathring{A}_n \mathring{R}_1^{n} \qquad \text{and} \qquad \mathring{A}_{-n} = \mathring{A}_n \dot{R}_1^{2n}$$

i.e.,

$$\mathring{A}_{-n} = \mathring{A}_n \left(\frac{\mathring{\mu}_1}{\mathring{\lambda}_1}\right)^n . \tag{33}$$

Similarly, if $\mathring{\alpha}_2 b < a$ then on the $\mathring{\zeta}_2$-plane one obtains

$$\mathring{B}_{-n} = \mathring{B}_n \mathring{R}_2^{2n} \qquad \text{or} \qquad \dot{B}_{-n} = \mathring{B}_n \left(\frac{\mathring{\mu}_2}{\mathring{\lambda}_2}\right)^n \tag{34}$$

where $\mathring{\mu}_1$, $\mathring{\mu}_2$, $\mathring{\lambda}_1$, $\mathring{\lambda}_2$ are constants for the disk analogous to (25) and (28).

Now if $\mathring{\alpha}_1 b > a$, the branch cut will lie along the y_1-axis; then as z_1 approaches the branch cut from the right side

$$\mathring{\zeta}_1 = \mathring{R}_3 e^{i\theta} \qquad \text{where} \quad -\pi/2 \leq \theta \leq \pi/2$$

and

$$\mathring{R}_3 = \sqrt{\frac{\mathring{\alpha}_1 b - a}{\mathring{\alpha}_1 b + a}} = \sqrt{\frac{\mathring{\mu}_1}{\mathring{\lambda}_1}} .$$

As z_1 approaches the branch cut from the left side

$$\mathring{\zeta}_1 = \mathring{R}_3\, e^{i\theta} \qquad \text{where } \pi/2 \le \theta \le -\pi/2$$

or

$$\mathring{\zeta}_1 = \mathring{R}_3\, e^{-i\theta} \qquad \text{where } -\pi/2 \le \theta \le \pi/2$$

If $\partial\mathring{\Phi}/\partial x$ is continuous as $\mathring{\zeta}_1$ approaches the branch cut from the right and from the left, then

$$\sum_{-\infty}^{\infty} \mathring{A}_n \mathring{R}_3^{\,n} e^{in\theta} = \sum_{-\infty}^{\infty} \mathring{A}_n \mathring{R}_3^{\,n} e^{-in\theta}$$

so that

$$\mathring{A}_{-n} \mathring{R}_3^{\,-n} = \mathring{A}_n \mathring{R}_3^{\,n} \qquad \text{or} \quad \mathring{A}_{-n} = \mathring{A}_n \mathring{R}_3^{\,2n}$$

And similarly for the $\mathring{\zeta}_2$-plane

$$\mathring{B}_{-n} = \mathring{B}_n \mathring{R}_4^{\,2n} \quad ,$$

these results being identical with (33) and (34).

General Equations of Displacement

For the plate (exterior of disk)

$$\Phi = R\left\{ F_1(x + i\alpha\epsilon y) + F_2\left(x + i\beta\epsilon y\right)\right\}$$

$$= R\left\{ F_1\left(x + i\alpha_1 y\right) + F_2\left(x + i\alpha_2 y\right)\right\}$$

$$= R\left\{ F_1(z_1) + F_2(z_2)\right\}$$

from which it will be convenient to set[10]

$$\frac{\partial \Phi}{\partial x} = R\left\{\frac{dF_1}{dz_1} + \frac{dF_2}{dz_2}\right\} = R\left\{\sum_{-\infty}^{1} A_n \varsigma_1^n + \sum_{-\infty}^{1} B_n \varsigma_2^n\right\} \tag{35}$$

where, from equation (29), it is seen that

$$\varsigma_1 = \frac{z_1 + \sqrt{z_1^2 - \lambda_1\mu_1}}{\lambda_1}$$

and

$$\varsigma_2 = \frac{z_2 + \sqrt{z_2^2 - \lambda_2\mu_2}}{\lambda_2}$$

Substituting the derivatives

$$\frac{d\varsigma_1}{dz_1} = \frac{2\varsigma_1^2}{\lambda_1\varsigma_1^2 - \mu_1}$$

and

$$\frac{d\varsigma_2}{dz_2} = \frac{2\varsigma_2^2}{\lambda_2\varsigma_2^2 - \mu_2} \tag{36}$$

one now obtains

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[10] Whereas it was found convenient to represent the interior of the disk by a Laurent series, it will now be desirable to represent the stress functions exterior to the disk by series of negative powers in ς inasmuch as the stresses in the plate at ∞ must approach constant values.

$$\sigma_y = \frac{\partial^2 \bar{\Phi}}{\partial x^2} = R\left\{ \frac{d^2 F_1}{d z_1^2} + \frac{d^2 F_2}{d z_2^2} \right\}$$

$$= 2R\left\{ \frac{A_1 \zeta_1^2}{\lambda_1 \zeta_1^2 - \mu_1} + \sum_{-\infty}^{-1} \frac{n A_n \zeta_1^{(n+1)}}{\lambda_1 \zeta_1^2 - \mu_1} + \frac{B_1 \zeta_2^2}{\lambda_2 \zeta_2^2 - \mu_2} + \sum_{-\infty}^{-1} \frac{n B_n \zeta_2^{(n+1)}}{\lambda_2 \zeta_2^2 - \mu_2} \right\} \quad (37)$$

Taking the partial derivative of $\bar{\Phi}$ with respect to y,

$$\frac{\partial \bar{\Phi}}{\partial y} = R\left\{ i\alpha_1 \frac{dF_1}{dz_1} + i\alpha_2 \frac{dF_2}{dz_2} \right\}$$

$$= R\left\{ i\alpha_1 \sum_{-\infty}^{1} A_n \zeta_1^n + i\alpha_2 \sum_{-\infty}^{1} B_n \zeta_2^n \right\} \quad (38)$$

so that

$$\sigma_x = \frac{\partial^2 \bar{\Phi}}{\partial y^2} = R\left\{ -\alpha_1^2 \frac{d^2 F_1}{d z_1^2} - \alpha_2^2 \frac{d^2 F_2}{d z_2^2} \right\}$$

$$= 2R\left\{ -\alpha_1^2 \left[\frac{A_1 \zeta_1^2}{\lambda_1 \zeta_1^2 - \mu_1} + \sum_{-\infty}^{-1} \frac{n A_n \zeta_1^{(n+1)}}{\lambda_1 \zeta_1^2 - \mu_1} \right] - \alpha_2^2 \left[\frac{B_1 \zeta_2^2}{\lambda_2 \zeta_2^2 - \mu_2} + \sum_{-\infty}^{-1} \frac{n B_n \zeta_2^{(n+1)}}{\lambda_2 \zeta_2^2 - \mu_2} \right] \right\} \quad (39)$$

and

$$\tau_{xy} = \frac{-\partial^2 \bar{\Phi}}{\partial x \, \partial y} = R\left\{ -i\alpha_1 \frac{d^2 F_1}{d z_1^2} - i\alpha_2 \frac{d^2 F_2}{d z_2^2} \right\}$$

$$= R\left\{ -i\alpha_1 \left[\frac{A_1 \zeta_1^2}{\lambda_1 \zeta_1^2 - \mu_1} + \sum_{-\infty}^{-1} \frac{n A_n \zeta_1^{(n+1)}}{\lambda_1 \zeta_1^2 - \mu_1} \right] - i\alpha_2 \left[\frac{B_1 \zeta_2^2}{\lambda_2 \zeta_2^2 - \mu_2} + \sum_{-\infty}^{-1} \frac{n B_n \zeta_2^{(n+1)}}{\lambda_2 \zeta_2^2 - \mu_2} \right] \right\}. \quad (40)$$

Now from (1) and (4)

$\frac{\partial u}{\partial x} = \epsilon_x = s_{11} \sigma_x + s_{12} \sigma_y$ and $\frac{\partial v}{\partial y} = \epsilon_y = s_{12} \sigma_x + s_{22} \sigma_y$.

Therefore

$$\frac{\partial u}{\partial x} = R\left\{(-\alpha_1^2 s_{11}+s_{12})\frac{d^2F_1}{dz_1^2} + (-\alpha_2^2 s_{11}+s_{12})\frac{d^2F_2}{dz_2^2}\right\} \tag{41}$$

and

$$\frac{\partial v}{\partial y} = R\left\{(-\alpha_1^2 s_{12}+s_{22})\frac{d^2F_1}{dz_1^2} + (-\alpha_2^2 s_{12}+s_{22})\frac{d^2F_2}{dz_2^2}\right\} \tag{42}$$

Integration of (41) and (42) yields

$$\begin{aligned} u &= R\left\{(-\alpha_1^2 s_{11}+s_{12})\frac{dF_1}{dz_1} + (-\alpha_2^2 s_{11}+s_{12})\frac{dF_2}{dz_2}\right\} + u_0(y) \\ v &= R\left\{\frac{(-\alpha_1^2 s_{12}+s_{22})}{i\alpha_1}\frac{dF_1}{dz_1} + \frac{(-\alpha_2^2 s_{12}+s_{22})}{i\alpha_2}\frac{dF_2}{dz_2}\right\} + v_0(x) \end{aligned} \tag{43}$$

Now from (1), (4) and (10)

$$\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = s_{33}\,\tau_{xy} \\ &= R\left\{s_{33}\left[-i\alpha_1\frac{d^2F_1}{dz_1^2} - i\alpha_2\frac{d^2F_2}{dz_2^2}\right]\right\} \end{aligned} \tag{44}$$

But from (43)

$$\frac{\partial u}{\partial y} = R\left\{i\alpha_1(-\alpha_1^2 s_{11}+s_{12})\frac{d^2F_1}{dz_1^2} + i\alpha_2(-\alpha_2^2 s_{11}+s_{12})\frac{d^2F_2}{dz_2^2}\right\} + u_0'(y)$$

and

$$\frac{\partial v}{\partial x} = R\left\{\frac{(-\alpha_1^2 s_{12}+s_{22})}{i\alpha_1}\frac{d^2F_1}{dz_1^2} + \frac{(-\alpha_2^2 s_{12}+s_{22})}{i\alpha_2}\frac{d^2F_2}{dz_2^2}\right\} + v_0'(x)$$

where $u_0'(y)$ and $v_0'(x)$ mean $\frac{\partial u_0(y)}{\partial y}$ and $\frac{\partial v_0(x)}{\partial x}$ respectively, so that

$$\begin{aligned} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = R\Bigg\{&\left[i\alpha_1(-\alpha_1^2 s_{11}+s_{12}) + \frac{(-\alpha_1^2 s_{12}+s_{22})}{i\alpha_1}\right]\frac{d^2F_1}{dz_1^2} \\ &+\left[i\alpha_2(-\alpha_2^2 s_{11}+s_{12}) + \frac{(-\alpha_2^2 s_{12}+s_{22})}{i\alpha_2}\right]\frac{d^2F_2}{dz_2^2}\Bigg\} + u_0'(y) + v_0'(x) \end{aligned} \tag{45}$$

i.e.,

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = R\left\{\left[i\alpha\epsilon\left(-\alpha^2\epsilon^2 s_{11} + s_{12}\right) + \left(\frac{-\alpha^2\epsilon^2 s_{12} + s_{22}}{i\alpha\epsilon}\right)\right]\frac{d^2F_1}{dz_1^2}\right.$$

$$+\left[i\beta\epsilon\left(-\beta^2\epsilon^2 s_{11} + s_{12}\right) + \left(\frac{-\beta^2\epsilon^2 s_{12} + s_{22}}{i\beta\epsilon}\right)\right]\frac{d^2F_2}{dz_2^2}$$

$$+ u_0'(y) + v_0'(x) \,. \qquad (46)$$

The first bracket of (46) reduces to

$$i\alpha\epsilon\left[-\alpha^2\epsilon^2 s_{11} + s_{12} + s_{12} - \frac{s_{22}}{\epsilon^2}\beta^2\right] .$$

Multiplying by ϵ^2/ϵ^2 this becomes

$$= \frac{i\alpha\epsilon}{\epsilon^2}\left[-\alpha^2\epsilon^4 s_{11} + 2s_{12}\epsilon^2 - s_{22}\beta^2\right] = \frac{i\alpha\epsilon}{\epsilon^2}\left[-\alpha^2\frac{s_{22}}{s_{11}}s_{11} - s_{22}\beta^2 + 2s_{12}\epsilon^2\right]$$

$$= \frac{i\alpha\epsilon}{\epsilon^2}\left[-s_{22}(\alpha^2+\beta^2) + 2s_{12}\epsilon^2\right] = \frac{i\alpha\epsilon}{\epsilon^2}\left[-2Ks_{22} + 2\epsilon^2 s_{12}\right]$$

$$= i\alpha\epsilon\left[\frac{-s_{22}}{\epsilon^2}\,\frac{2s_{12}+s_{33}}{\sqrt{s_{11}s_{22}}} + 2s_{12}\right]$$

$$= i\alpha\epsilon\left[\frac{-s_{22}}{\sqrt{s_{11}s_{22}}}\sqrt{\frac{s_{11}}{s_{22}}}\left(2s_{12}+s_{33}\right) + 2s_{12}\right]$$

$$= i\alpha\epsilon\left[-\left(2s_{12}+s_{33}\right) + 2s_{12}\right] = i\alpha\epsilon\left(-s_{33}\right)$$

where it is recalled from (12) that

$$\epsilon^4 = \frac{s_{22}}{s_{11}} \quad , \qquad \alpha^2 + \beta^2 = 2K$$

$$\beta = \frac{1}{\alpha} \quad , \qquad K = \frac{1}{2\sqrt{s_{11}s_{22}}}\left(2s_{12} + s_{33}\right) .$$

Similarly, the second bracket in (46), viz.,

$$\left[i\beta\epsilon\left(-\beta^2\epsilon^2 s_{11} + s_{12}\right) + \left(\frac{-\beta^2\epsilon^2 s_{12} + s_{22}}{i\beta\epsilon}\right)\right]$$

simplifies to

$$i\beta\epsilon\left(-s_{33}\right) .$$

Replacing the brackets of (45) with these expressions and then equating (44) and (45) reveals that

$$u_o'(y) + v_o'(x) = 0$$

i.e.,

$$\frac{\partial u_o(y)}{\partial y} + \frac{\partial v_o(x)}{\partial x} = 0$$

Since the left hand member in this equation is a function of y only, while the right hand member is a function of x only, we are lead to conclude that both members must be equal to some constant, call it C; then

$$\frac{\partial u_o(y)}{\partial y} = C$$

and

$$\frac{\partial v_o(x)}{\partial x} = -C$$

Hence

$$\frac{\partial u}{\partial y} = R\left\{ i\alpha\epsilon\left(-\alpha^2\epsilon^2 s_{11}+s_{12}\right)\frac{d^2F_1}{dz_1^2} + i\beta\epsilon\left(-\beta^2\epsilon^2 s_{11}+s_{12}\right)\frac{d^2F_2}{dz_2^2} \right\} + C \tag{47}$$

and

$$\frac{\partial v}{\partial x} = R\left\{ \frac{\left(-\alpha^2\epsilon^2 s_{12}+s_{22}\right)}{i\alpha\epsilon}\frac{d^2F_1}{dz_1^2} + \frac{\left(-\beta^2\epsilon^2 s_{12}+s_{22}\right)}{i\beta\epsilon}\frac{d^2F_2}{dz_2^2} \right\} - C \tag{48}$$

Integrating (47) and (48),

$$u = R\left\{ \left(-\alpha^2\epsilon^2 s_{11}+s_{12}\right)\frac{dF_1}{dz_1} + \left(-\beta^2\epsilon^2 s_{11}+s_{12}\right)\frac{dF_2}{dz_2} \right\} + Cy + D$$

$$v = R\left\{ \frac{\left(-\alpha^2\epsilon^2 s_{12}+s_{22}\right)}{i\alpha\epsilon}\frac{dF_1}{dz_1} + \frac{\left(-\beta^2\epsilon^2 s_{12}+s_{22}\right)}{i\beta\epsilon}\frac{dF_2}{dz_2} \right\} - Cx + E$$

that is

$$\begin{aligned} u &= R\left\{ \left(-\alpha_1^2 s_{11}+s_{12}\right)\sum_{-\infty}^{1} A_n \zeta_1^n + \left(-\alpha_2^2 s_{11}+s_{12}\right)\sum_{-\infty}^{1} B_n \zeta_2^n \right\} + Cy + D \\ v &= R\left\{ \frac{\left(-\alpha_1^2 s_{12}+s_{22}\right)}{i\alpha_1}\sum_{-\infty}^{1} A_n \zeta_1^n + \frac{\left(-\alpha_2^2 s_{12}+s_{22}\right)}{i\alpha_2}\sum_{-\infty}^{1} B_n \zeta_2^n \right\} - Cx + E \end{aligned} \tag{49}$$

Similarly for the interior of the disk we obtain

$$\begin{aligned} \mathring{u} &= R\left\{ \left(-\mathring{\alpha}_1^2 \mathring{s}_{11}+\mathring{s}_{12}\right)\sum_{-\infty}^{\infty} \mathring{A}_n \mathring{\zeta}_1^n + \left(-\mathring{\alpha}_2^2 \mathring{s}_{11}+\mathring{s}_{12}\right)\sum_{-\infty}^{\infty} \mathring{B}_n \mathring{\zeta}_2^n \right\} + Cy + D \\ \mathring{v} &= R\left\{ \frac{\left(-\mathring{\alpha}_1^2 \mathring{s}_{12}+\mathring{s}_{22}\right)}{i\mathring{\alpha}_1}\sum_{-\infty}^{\infty} \mathring{A}_n \mathring{\zeta}_1^n + \frac{\left(-\mathring{\alpha}_2^2 \mathring{s}_{12}+\mathring{s}_{22}\right)}{i\mathring{\alpha}_2}\sum_{-\infty}^{\infty} \mathring{B}_n \mathring{\zeta}_2^n \right\} - Cx + E \end{aligned} \tag{50}$$

The linear functions of x and y in equations (49) and (50) can be disposed of by fixing the origin and eliminating rotation and translation with respect to that origin. If this is done, the expressions for u , v , $\mathring{u}$ and $\mathring{v}$ reduce to the terms within the respective braces; i.e.,

$$u = R\left\{\left(-\alpha_1^2 s_{11}+s_{12}\right)\sum_{-\infty}^{1} A_n \zeta_1^n + \left(-\alpha_2^2 s_{11}+s_{12}\right)\sum_{-\infty}^{1} B_n \zeta_2^n\right\}$$

$$v = R\left\{\frac{\left(-\alpha_1^2 s_{12}+s_{22}\right)}{i\alpha_1}\sum_{-\infty}^{1} A_n \zeta_1^n + \frac{\left(-\alpha_2^2 s_{12}+s_{22}\right)}{i\alpha_2}\sum_{-\infty}^{1} B_n \zeta_2^n\right\} \tag{51}$$

$$\mathring{u} = R\left\{\left(-\mathring{\alpha}_1^2 \mathring{s}_{11}+\mathring{s}_{12}\right)\sum_{-\infty}^{\infty} \mathring{A}_n \mathring{\zeta}_1^n + \left(-\mathring{\alpha}_2^2 \mathring{s}_{11}+\mathring{s}_{12}\right)\sum_{-\infty}^{\infty} \mathring{B}_n \mathring{\zeta}_2^n\right\}$$

$$\mathring{v} = R\left\{\frac{\left(-\mathring{\alpha}_1^2 \mathring{s}_{12}+\mathring{s}_{22}\right)}{i\mathring{\alpha}_1}\sum_{-\infty}^{\infty} \mathring{A}_n \mathring{\zeta}_1^n + \frac{\left(-\mathring{\alpha}_2^2 \mathring{s}_{12}+\mathring{s}_{22}\right)}{i\mathring{\alpha}_2}\sum_{-\infty}^{\infty} \mathring{B}_n \mathring{\zeta}_2^n\right\}$$

Determination of Coefficients

The complex coefficients in the series representing the stress functions can now be determined by imposing given boundary restrictions on the disk and on the plate.

Since a uniform tension S is acting upon two opposite edges of the infinite plate, and since ζ exhibits the same behavior at infinity as z, then it is seen from (22), (37), (39) and (40) that

$$\lim_{\substack{z_1 \to \infty \\ z_2 \to \infty}} \sigma_x = R\left\{\frac{-2\alpha_1^2 A_1}{\lambda_1} - \frac{2\alpha_2^2 B_1}{\lambda_2}\right\} = S \qquad (52)$$

$$\lim_{\substack{z_1 \to \infty \\ z_2 \to \infty}} \sigma_y = R\left\{\frac{2A_1}{\lambda_1} + \frac{2B_1}{\lambda_2}\right\} = 0 \qquad (53)$$

and

$$\lim_{\substack{z_1 \to \infty \\ z_2 \to \infty}} \tau_{xy} = R\left\{\frac{i\alpha_1 A_1}{\lambda_1} - \frac{i\alpha_2 B_1}{\lambda_2}\right\} = 0 \ . \qquad (54)$$

Now from (52)

$$\frac{\alpha_1^2 A_1'}{\lambda_1} + \frac{\alpha_2^2 B_1'}{\lambda_2} = \frac{-S}{2}$$

and from (53)

$$\frac{A_1'}{\lambda_1} + \frac{B_1'}{\lambda_2} = 0$$

where the primes denote the real parts of the complex coefficients.

Therefore

$$B_1' = \frac{-\lambda_2}{\lambda_1} A_1'$$

and

$$A_1' = \frac{S\lambda_1}{2(\alpha_2^2 - \alpha_1^2)}$$

$$B_1' = \frac{-S\lambda_2}{2(\alpha_2^2 - \alpha_1^2)} \tag{55}$$

From (54)

$$\frac{\alpha_1 A_1''}{\lambda_1} + \frac{\alpha_2 B_1''}{\lambda_2} = 0$$

Therefore

$$B_1'' = \frac{-\alpha_1 \lambda_2}{\alpha_2 \lambda_1} A_1'' \tag{56}$$

where the double primes denote the imaginary parts of the complex coefficients.

Now at the boundary of the disk the ellipse has been mapped into a unit circle so that here

$$\zeta_1 = \zeta_2 = \mathring{\zeta}_1 = \mathring{\zeta}_2 = e^{i\theta} \tag{57}$$

Making this substitution in the first condition of (21), i.e., setting $\mathring{u} = u$ yields

$$R\left\{(-\mathring{\alpha}_1^2 \mathring{s}_{11} + \mathring{s}_{12}) \sum_{-\infty}^{\infty} \mathring{A}_n e^{in\theta} + (-\mathring{\alpha}_2^2 \mathring{s}_{11} + \mathring{s}_{12}) \sum_{-\infty}^{\infty} \mathring{B}_n e^{in\theta}\right\}$$

$$= R\left\{(-\alpha_1^2 s_{11} + s_{12}) \sum_{-\infty}^{1} A_n e^{in\theta} + (-\alpha_2^2 s_{11} + s_{12}) \sum_{-\infty}^{1} B_n e^{in\theta}\right\}$$

where the general expressions for u and $\mathring{u}$ were given in equations (51).

The preceding equality can be written as

$$R\left\{ \sum_{-\infty}^{\infty} \mathring{c}_1 \mathring{A}_n e^{in\theta} + \sum_{-\infty}^{\infty} \mathring{c}_2 \mathring{B}_n e^{in\theta} \right\}$$

$$= R\left\{ \sum_{-\infty}^{1} c_1 A_n e^{in\theta} + \sum_{-\infty}^{1} c_2 B_n e^{in\theta} \right\} \tag{58}$$

where

$$c_1 = \left(-\alpha_1^2 s_{11} + s_{12}\right) , \quad \mathring{c}_1 = \left(-\mathring{\alpha}_1^2 \mathring{s}_{11} + \mathring{s}_{12}\right)$$

$$c_2 = \left(-\alpha_2^2 s_{11} + s_{12}\right) , \quad \mathring{c}_2 = \left(-\mathring{\alpha}_2^2 \mathring{s}_{11} + \mathring{s}_{12}\right) \tag{59}$$

Similarly from $\mathring{v} = v$ is obtained the relationship

$$R\left\{ \sum_{-\infty}^{\infty} \mathring{d}_1 \left(-i \mathring{A}_n e^{in\theta}\right) + \sum_{-\infty}^{\infty} \mathring{d}_2 \left(-i \mathring{B}_n e^{in\theta}\right) \right\}$$

$$= R\left\{ \sum_{-\infty}^{1} d_1 \left(-i A_n e^{in\theta}\right) + \sum_{-\infty}^{1} d_2 \left(-i B_n e^{in\theta}\right) \right\} \tag{60}$$

where

$$d_1 = \left(\frac{-\alpha_1^2 s_{12} + s_{22}}{\alpha_1}\right) , \quad \mathring{d}_1 = \left(\frac{-\mathring{\alpha}_1^2 \mathring{s}_{12} + \mathring{s}_{22}}{\mathring{\alpha}_1}\right)$$

$$d_2 = \left(\frac{-\alpha_2^2 s_{12} + s_{22}}{\alpha_2}\right) , \quad \mathring{d}_2 = \left(\frac{-\mathring{\alpha}_2^2 \mathring{s}_{12} + \mathring{s}_{22}}{\mathring{\alpha}_2}\right) \tag{61}$$

Dividing out a two, equations (58) and (60) can be written as

$$\sum_{-\infty}^{\infty} \left[\mathring{c}_1 \left(\mathring{A}_n e^{in\theta} + \overline{\mathring{A}}_n e^{-in\theta} \right) + \mathring{c}_2 \left(\mathring{B}_n e^{in\theta} + \overline{\mathring{B}}_n e^{-in\theta} \right) \right]$$

$$= \sum_{-\infty}^{1} \left[c_1 \left(A_n e^{in\theta} + \overline{A}_n e^{-in\theta} \right) + c_2 \left(B_n e^{in\theta} + \overline{B}_n e^{-in\theta} \right) \right] \tag{62}$$

and

$$\sum_{-\infty}^{\infty} \left[\mathring{d}_1 \left(-i \mathring{A}_n e^{in\theta} + i \overline{\mathring{A}}_n e^{-in\theta} \right) + \mathring{d}_2 \left(-i \mathring{B}_n e^{in\theta} + i \overline{\mathring{B}}_n e^{-in\theta} \right) \right]$$

$$= \sum_{-\infty}^{1} \left[d_1 \left(-i A_n e^{in\theta} + i \overline{A}_n e^{-in\theta} \right) + d_2 \left(-i B_n e^{in\theta} + i \overline{B}_n e^{-in\theta} \right) \right] \tag{63}$$

from which it is seen that

$$\mathring{c}_1\left(\mathring{A}_n+\bar{\mathring{A}}_{-n}\right)+\mathring{c}_2\left(\mathring{B}_n+\bar{\mathring{B}}_{-n}\right)=c_1\left(A_n+\bar{A}_{-n}\right)+c_2\left(B_n+\bar{B}_{-n}\right) \tag{64}$$

and

$$\begin{aligned}\mathring{d}_1\left(-i\mathring{A}_n+i\bar{\mathring{A}}_{-n}\right)&+\mathring{d}_2\left(-i\mathring{B}_n+i\bar{\mathring{B}}_{-n}\right)\\ &=d_1\left(-iA_n+i\bar{A}_{-n}\right)+d_2\left(-iB_n+i\bar{B}_{-n}\right)\end{aligned} \tag{65}$$

where $n = 0, 1, 2, \ldots$ and $A_n = B_n = 0$ for $n \geq 2$. Now since $\mathring{A}_{-n} = \mathring{A}_n(\mathring{R}_1)^{2n}$ and $\mathring{B}_{-n} = \mathring{B}_n(\mathring{R}_2)^{2n}$ by equations (33) and (34), then (64) and (65) become

$$\begin{aligned}\mathring{c}_1\left[\mathring{A}_n+\bar{\mathring{A}}_n\left(\mathring{R}_1\right)^{2n}\right]&+\mathring{c}_2\left[\mathring{B}_n+\bar{\mathring{B}}_n\left(\mathring{R}_2\right)^{2n}\right]\\ &=c_1\left(A_n+\bar{A}_{-n}\right)+c_2\left(B_n+\bar{B}_{-n}\right)\end{aligned} \tag{66}$$

$$\begin{aligned}i\mathring{d}_1\left[-\mathring{A}_n+\bar{\mathring{A}}_n\left(\mathring{R}_1\right)^{2n}\right]&+i\mathring{d}_2\left[-\mathring{B}_n+\bar{\mathring{B}}_n\left(\mathring{R}_2\right)^{2n}\right]\\ &=id_1\left(-A_n+\bar{A}_{-n}\right)+id_2\left(-B_n+\bar{B}_{-n}\right)\end{aligned} \tag{67}$$

for the values $n = 0, 1$.

For $n \geq 2$, these equations reduce to

$$\mathring{c}_1\left[\mathring{A}_n+\bar{\mathring{A}}_n\left(\mathring{R}_1\right)^{2n}\right]+\mathring{c}_2\left[\mathring{B}_n+\bar{\mathring{B}}_n\left(\mathring{R}_2\right)^{2n}\right]=c_1\bar{A}_{-n}+c_2\bar{B}_{-n} \tag{68}$$

and

$$\begin{aligned}i\mathring{d}_1\left[-\mathring{A}_n+\bar{\mathring{A}}_n\left(\mathring{R}_1\right)^{2n}\right]&+i\mathring{d}_2\left[-\mathring{B}_n+\bar{\mathring{B}}_n\left(\mathring{R}_2\right)^{2n}\right]\\ &=id_1\bar{A}_{-n}+id_2\bar{B}_{-n}\end{aligned} \tag{69}$$

Now on the boundary of the disk the real part of the function $\mathring{\Phi}$ satisfies the relations

$$\frac{d}{ds}\left(\frac{\partial\mathring{\Phi}}{\partial y}\right)=\mathring{X}_\nu \quad , \quad \frac{d}{ds}\left(\frac{\partial\mathring{\Phi}}{\partial x}\right)=-\mathring{Y}_\nu$$

where ds denotes an element of the boundary and $\mathring{X}_\nu\,ds$ and $\mathring{Y}_\nu\,ds$ represent the x- and y-components of the force per unit thickness of the disk acting on the element ds from the positive side as indicated by the direction of ν, the normal to the boundary (cf. figure 3). Similarly, in the plate

$$\frac{d}{ds}\left(\frac{\partial\Phi}{\partial y}\right)=X_\nu \quad , \quad \frac{d}{ds}\left(\frac{\partial\Phi}{\partial x}\right)=-Y_\nu$$

But the boundary of the ellipse has been mapped into a unit circle so that

$$\frac{d}{ds}\left(\frac{\partial\Phi}{\partial x}\right)=\frac{d}{d\theta}\left(\frac{\partial\Phi}{\partial x}\right)\frac{d\theta}{ds}=\frac{d}{d\theta}\left(\frac{\partial\Phi}{\partial x}\right)\frac{1}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

or

$$\frac{d}{d\theta}\left(\frac{\partial\Phi}{\partial x}\right)\frac{1}{\pi}=-Y_\nu \qquad \text{and} \quad \frac{d}{d\theta}\left(\frac{\partial\mathring{\Phi}}{\partial x}\right)\frac{1}{\pi}=-\mathring{Y}_\nu$$

where

$$\pi=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

Thus to preserve continuity of the function in going from the plate to the disk the following relationships are required at the boundary:

$$\frac{d}{d\theta}\left(\frac{\partial\Phi}{\partial x}\right)=\frac{d}{d\theta}\left(\frac{\partial\mathring{\Phi}}{\partial x}\right) \tag{70}$$

and

$$\frac{d}{d\theta}\left(\frac{\partial\Phi}{\partial y}\right)=\frac{d}{d\theta}\left(\frac{\partial\mathring{\Phi}}{\partial y}\right) \tag{71}$$

Now

$$\frac{d}{d\theta}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{d}{d\theta}\, R\left\{\sum_{-\infty}^{1} A_n e^{in\theta} + \sum_{-\infty}^{1} B_n e^{in\theta}\right\}$$

$$= \tfrac{1}{2}\frac{d}{d\theta}\left[\sum_{-\infty}^{1}\left(A_n e^{in\theta} + \bar{A}_n e^{-in\theta}\right) + \sum_{-\infty}^{1}\left(B_n e^{in\theta} + \bar{B}_n e^{-in\theta}\right)\right]$$

$$= \frac{i}{2}\left[\sum_{-\infty}^{1} n\left(A_n e^{in\theta} + \bar{A}_{-n} e^{in\theta}\right) + \sum_{-\infty}^{1} n\left(B_n e^{in\theta} + \bar{B}_{-n} e^{in\theta}\right)\right] \qquad (72)$$

Also

$$\frac{d}{d\theta}\left(\frac{\partial \mathring{\Phi}}{\partial x}\right) = \frac{i}{2}\left\{\sum_{-\infty}^{\infty} n\left[\mathring{A}_n + \bar{\mathring{A}}_n (\mathring{R}_1)^{2n}\right] e^{in\theta} + \sum_{-\infty}^{\infty} n\left[\mathring{B}_n + \bar{\mathring{B}}_n (\mathring{R}_2)^{2n}\right] e^{in\theta}\right\} \qquad (73)$$

Similarly

$$\frac{d}{d\theta}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{-\alpha_1}{2}\sum_{-\infty}^{1} n\left(A_n - \bar{A}_{-n}\right) e^{in\theta} - \frac{\alpha_2}{2}\sum_{-\infty}^{1} n\left(B_n - \bar{B}_{-n}\right) e^{in\theta} \qquad (74)$$

and

$$\frac{d}{d\theta}\left(\frac{\partial \mathring{\Phi}}{\partial y}\right) = \frac{-\mathring{\alpha}_1}{2}\sum_{-\infty}^{\infty} n\left[\mathring{A}_n - \bar{\mathring{A}}_n (\mathring{R}_1)^{2n}\right] e^{in\theta} - \frac{\mathring{\alpha}_2}{2}\sum_{-\infty}^{\infty} n\left[\mathring{B}_n - \bar{\mathring{B}}_n (\mathring{R}_2)^{2n}\right] e^{in\theta} \qquad (75)$$

Setting (73) equal to (72) yields the equations

$$\left[\mathring{A}_n + \bar{\mathring{A}}_n (\mathring{R}_1)^{2n}\right] + \left[\mathring{B}_n + \bar{\mathring{B}}_n (\mathring{R}_2)^{2n}\right] = A_n + \bar{A}_{-n} + B_n + \bar{B}_{-n} \qquad \text{for } n = 1 \quad (76)$$

and

$$\left[\mathring{A}_n + \bar{\mathring{A}}_n (\mathring{R}_1)^{2n}\right] + \left[\mathring{B}_n + \bar{\mathring{B}}_n (\mathring{R}_2)^{2n}\right] = \bar{A}_{-n} + \bar{B}_{-n} \qquad \text{for } n \geq 2. \quad (77)$$

Similarly, from (75) and (74) we have

$$\mathring{\alpha}_1\left[\mathring{A}_n - \bar{\mathring{A}}_n (\mathring{R}_1)^{2n}\right] + \mathring{\alpha}_2\left[\mathring{B}_n - \bar{\mathring{B}}_n (\mathring{R}_2)^{2n}\right]$$

$$= \alpha_1\left[A_n - \bar{A}_{-n}\right] + \alpha_2\left[B_n - \bar{B}_{-n}\right] \qquad (78)$$

for n = 1, and

$$\mathring{\alpha}_1\left[\mathring{A}_n - \bar{\mathring{A}}_n(\mathring{R}_1)^{2n}\right] + \mathring{\alpha}_2\left[\mathring{B}_n - \bar{\mathring{B}}_n(\mathring{R}_2)^{2n}\right] = -\alpha_1 \bar{A}_{-n} - \alpha_2 \bar{B}_{-n} \tag{79}$$

for $n \geq 2$. Now after taking conjugates there remain, for the case $n \geq 2$, the four equations (68), (69), (77) and (79) in the unknowns $\mathring{A}'_n$, $\mathring{B}'_n$, A'_{-n}, B'_{-n}, and similarly for $\mathring{A}''_n$, $\mathring{B}''_n$, A''_{-n} and B''_{-n} ; i.e., for $n \geq 2$,

$$\begin{aligned}
&\mathring{c}_1\left[\mathring{A}'_n + \mathring{A}'_n(\mathring{R}_1)^{2n}\right] + \mathring{c}_2\left[\mathring{B}'_n + \mathring{B}'_n(\mathring{R}_2)^{2n}\right] = c_1 A'_{-n} + c_2 B'_{-n}\\
&\mathring{d}_1\left[\mathring{A}'_n - \mathring{A}'_n(\mathring{R}_1)^{2n}\right] + \mathring{d}_2\left[\mathring{B}'_n - \mathring{B}'_n(\mathring{R}_2)^{2n}\right] = -d_1 A'_{-n} - d_2 B'_{-n}\\
&\left[\mathring{A}'_n + \mathring{A}'_n(\mathring{R}_1)^{2n}\right] + \left[\mathring{B}'_n + \mathring{B}'_n(\mathring{R}_2)^{2n}\right] = A'_{-n} + B'_{-n}\\
&\mathring{\alpha}_1\left[\mathring{A}'_n - \mathring{A}'_n(\mathring{R}_1)^{2n}\right] + \mathring{\alpha}_2\left[\mathring{B}'_n - \mathring{B}'_n(\mathring{R}_2)^{2n}\right] = -\alpha_1 A'_{-n} - \alpha_2 B'_{-n}
\end{aligned} \tag{80}$$

and four similar equations involving the imaginary coefficients $\mathring{A}''_n$, $\mathring{B}''_n$, A''_{-n} and B''_{-n} , viz.,

$$\begin{aligned}
&\mathring{c}_1\left(1 + \mathring{R}_1^{2n}\right)\mathring{A}''_n + \mathring{c}_2\left(1 + \mathring{R}_2^{2n}\right)\mathring{B}''_n = c_1 A''_{-n} + c_2 B''_{-n}\\
&\mathring{d}_1\left(1 - \mathring{R}_1^{2n}\right)\mathring{A}''_n + \mathring{d}_2\left(1 - \mathring{R}_2^{2n}\right)\mathring{B}''_n = -d_1 A''_{-n} - d_2 B''_{-n}\\
&\left(1 + \mathring{R}_1^{2n}\right)\mathring{A}''_n + \left(1 + \mathring{R}_2^{2n}\right)\mathring{B}''_n = A''_{-n} + B''_{-n}\\
&\mathring{\alpha}_1\left(1 - \mathring{R}_1^{2n}\right)\mathring{A}''_n + \mathring{\alpha}_2\left(1 - \mathring{R}_2^{2n}\right)\mathring{B}''_n = -\alpha_1 A''_{-n} - \alpha_2 B''_{-n}
\end{aligned} \tag{81}$$

Let the preceding equations (81) be written as

$$\mathring{c}_1 L \mathring{A}''_n + \mathring{c}_2 M \mathring{B}''_n - c_1 A''_{-n} - c_2 B''_{-n} = 0$$
$$\mathring{d}_1 P \mathring{A}''_n + \mathring{d}_2 Q \mathring{B}''_n + d_1 A''_{-n} + d_2 B''_{-n} = 0$$
$$L \mathring{A}''_n + M \mathring{B}''_n - A''_{-n} - B''_{-n} = 0$$
$$\mathring{\alpha}_1 P \mathring{A}''_n + \mathring{\alpha}_2 Q \mathring{B}''_n + \alpha_1 A''_{-n} + \alpha_2 B''_{-n} = 0$$

where

$$L = 1 + (\mathring{R}_1)^{2n} \qquad P = 1 - (\mathring{R}_1)^{2n}$$
$$M = 1 + (\mathring{R}_2)^{2n} \qquad Q = 1 - (\mathring{R}_2)^{2n} \; .$$

Then it is seen from the above set of four homogeneous equations in four unknowns that

$$\mathring{A}''_n = \mathring{B}''_n = A''_{-n} = B''_{-n} = 0 \qquad \text{for} \quad n \geq 2$$

and similarly from (80)

$$\mathring{A}'_n = \mathring{B}'_n = A'_{-n} = B'_{-n} = 0 \qquad \text{for} \quad n \geq 2 \, .$$

Since the coefficients $\mathring{A}_n$, A_{-n}, etc, for the case $n = 0$ do not enter into the expressions for the stresses or strains, this case can be ignored so that there now remain equations (66), (67), (76) and (78) for determining coefficients in which n takes the value unity; i.e.,

$$\mathring{c}_1\left[\mathring{A}_1 + \bar{\mathring{A}}_1(\mathring{R}_1)^2\right] + \mathring{c}_2\left[\mathring{B}_1 + \bar{\mathring{B}}_1(\mathring{R}_2)^2\right] = c_1\left(A_1 + \bar{A}_{-1}\right) + c_2\left(B_1 + \bar{B}_{-1}\right)$$
$$\mathring{d}_1\left[\mathring{A}_1 - \bar{\mathring{A}}_1(\mathring{R}_1)^2\right] + \mathring{d}_2\left[\mathring{B}_1 - \bar{\mathring{B}}_1(\mathring{R}_2)^2\right] = d_1\left(A_1 - \bar{A}_{-1}\right) + d_2\left(B_1 - \bar{B}_{-1}\right) \qquad (82)$$
$$\left[\mathring{A}_1 + \bar{\mathring{A}}_1(\mathring{R}_1)^2\right] + \left[\mathring{B}_1 + \bar{\mathring{B}}_1(\mathring{R}_2)^2\right] = \left(A_1 + \bar{A}_{-1}\right) + \left(B_1 + \bar{B}_{-1}\right)$$
$$\mathring{\alpha}_1\left[\mathring{A}_1 - \bar{\mathring{A}}_1(\mathring{R}_1)^2\right] + \mathring{\alpha}_2\left[\mathring{B}_1 - \bar{\mathring{B}}_1(\mathring{R}_2)^2\right] = \alpha_1\left(A_1 - \bar{A}_{-1}\right) + \alpha_2\left(B_1 - \bar{B}_{-1}\right)$$

Taking the conjugates of (82) and successively adding and subtracting yields the two sets of equations

$$\begin{aligned}
&\mathring{c}_1(1+\mathring{R}_1^2)\mathring{A}_1' + \mathring{c}_2(1+\mathring{R}_2^2)\mathring{B}_1' = c_1(A_1'+A_{-1}') + c_2(B_1'+B_{-1}') \\
&\mathring{d}_1(1-\mathring{R}_1^2)\mathring{A}_1' + \mathring{d}_2(1-\mathring{R}_2^2)\mathring{B}_1' = d_1(A_1'-A_{-1}') + d_2(B_1'-B_{-1}') \\
&(1+\mathring{R}_1^2)\mathring{A}_1' + (1+\mathring{R}_2^2)\mathring{B}_1' = A_1'+A_{-1}'+B_1'+B_{-1}' \\
&\mathring{\alpha}_1(1-\mathring{R}_1^2)\mathring{A}_1' + \mathring{\alpha}_2(1-\mathring{R}_2^2)\mathring{B}_1' = \alpha_1(A_1'-A_{-1}') + \alpha_2(B_1'-B_{-1}')
\end{aligned} \tag{83}$$

and

$$\begin{aligned}
&\mathring{c}_1(1-\mathring{R}_1^2)\mathring{A}_1'' + \mathring{c}_2(1-\mathring{R}_2^2)\mathring{B}_1'' = c_1(A_1''-A_{-1}'') + c_2(B_1''-B_{-1}'') \\
&\mathring{d}_1(1+\mathring{R}_1^2)\mathring{A}_1'' + \mathring{d}_2(1+\mathring{R}_2^2)\mathring{B}_1'' = d_1(A_1''+A_{-1}'') + d_2(B_1''+B_{-1}'') \\
&(1-\mathring{R}_1^2)\mathring{A}_1'' + (1-\mathring{R}_2^2)\mathring{B}_1'' = A_1''-A_{-1}''+B_1''-B_{-1}'' \\
&\mathring{\alpha}_1(1+\mathring{R}_1^2)\mathring{A}_1'' + \mathring{\alpha}_2(1+\mathring{R}_2^2)\mathring{B}_1'' = \alpha_1(A_1''+A_{-1}'') + \alpha_2(B_1''+B_{-1}'')
\end{aligned} \tag{84}$$

Equations (84) and (56) involve five equations but six unknown coefficients. Therefore we shall try setting these imaginary parts equal to zero; i.e., set

$$\mathring{A}_1'' = \mathring{B}_1'' = A_{-1}'' = B_{-1}'' = 0 .$$

If this is done, then the stresses will involve only real coefficients which can be obtained from (83).

To solve (83) rewrite it as follows:

$$\begin{aligned}
&\mathring{c}_1 L \mathring{A}_1' + \mathring{c}_2 M \mathring{B}_1' - c_1 A_{-1}' - c_2 B_{-1}' = (c_1 A_1' + c_2 B_1') \\
&\mathring{d}_1 P \mathring{A}_1' + \mathring{d}_2 Q \mathring{B}_1' + d_1 A_{-1}' + d_2 B_{-1}' = (d_1 A_1' + d_2 B_1') \\
&\quad L \mathring{A}_1' + M \mathring{B}_1' - A_{-1}' - B_{-1}' = (A_1' + B_1') \\
&\mathring{\alpha}_1 P \mathring{A}_1' + \mathring{\alpha}_2 Q \mathring{B}_1' + \alpha_1 A_{-1}' + \alpha_2 B_{-1}' = (\alpha_1 A_1' + \alpha_2 B_1')
\end{aligned} \tag{85}$$

where now

$$L = 1 + (\mathring{R}_1)^2 = 2a/\lambda_1^\circ \;, \quad P = 1 - (\mathring{R}_1)^2 = 2\mathring{\alpha}_1 b/\lambda_1^\circ$$
$$M = 1 + (\mathring{R}_2)^2 = 2a/\lambda_2^\circ \;, \quad Q = 1 - (\mathring{R}_2)^2 = 2\mathring{\alpha}_2 b/\lambda_2^\circ \;.$$

The determinant Δ of the coefficients is

$$\Delta = \begin{vmatrix} \mathring{c}_1 L & \mathring{c}_2 M & -c_1 & -c_2 \\ \mathring{d}_1 P & \mathring{d}_2 Q & d_1 & d_2 \\ L & M & -1 & -1 \\ \mathring{\alpha}_1 P & \mathring{\alpha}_2 Q & \alpha_1 & \alpha_2 \end{vmatrix}$$

Expanding and then substituting the above values for L, M, P and Q leads to the expression

$$\begin{aligned}
\Delta = \frac{4}{\lambda_1^\circ \lambda_2^\circ} \Big\{ & a^2\big[(\mathring{c}_1 - \mathring{c}_2)(d_2\alpha_1 - d_1\alpha_2)\big] + a\mathring{\alpha}_1 b\big[\mathring{c}_2\mathring{d}_1(\alpha_2 - \alpha_1) \\
& + \mathring{c}_2\mathring{\alpha}_1(d_1 - d_2) + \mathring{d}_1(c_2\alpha_1 - c_1\alpha_2) + \mathring{\alpha}_1(c_1 d_2 - c_2 d_1)\big] \\
& + a\mathring{\alpha}_2 b\big[\mathring{c}_1\mathring{d}_2(\alpha_1 - \alpha_2) - \mathring{c}_1\mathring{\alpha}_2(d_1 - d_2) + \mathring{d}_2(c_1\alpha_2 - c_2\alpha_1) \\
& + \mathring{\alpha}_2(c_2 d_1 - c_1 d_2)\big] + \mathring{\alpha}_1\mathring{\alpha}_2 b^2 (c_1 - c_2)(\mathring{d}_2\mathring{\alpha}_1 - \mathring{d}_1\mathring{\alpha}_2) \Big\} \;.
\end{aligned} \tag{86}$$

Now for the product $\Delta\ \mathring{A}_1'$ we have the determinant

$$\Delta\ \mathring{A}_1' = \begin{vmatrix} (c_1 A_1' + c_2 B_1') & \mathring{c}_2 M & -c_1 & -c_2 \\ (d_1 A_1' + d_2 B_1') & \mathring{d}_2 Q & d_1 & d_2 \\ (A_1' + B_1') & M & -1 & -1 \\ (\alpha_1 A_1' + \alpha_2 B_1') & \mathring{\alpha}_2 Q & \alpha_1 & \alpha_2 \end{vmatrix}$$

Expanding in terms of A_1' and B_1' and then making substitutions from equations (55) leads to the expression

$$\begin{aligned} \Delta\ \mathring{A}_1' = \frac{2S}{\mathring{\lambda}_2(\alpha_2^2 - \alpha_1^2)} \Big\{ & a\lambda_1(c_1 d_2 \alpha_1 - c_1 d_1 \alpha_2 + \mathring{c}_2 d_1 \alpha_2 - \mathring{c}_2 d_2 \alpha_1) \\ & + a\lambda_2(c_2 d_1 \alpha_2 - c_2 d_2 \alpha_1 + \mathring{c}_2 d_2 \alpha_1 - \mathring{c}_2 d_1 \alpha_2) \\ & + \mathring{\alpha}_2 b\lambda_1(c_1 \mathring{d}_2 \alpha_1 - c_2 \mathring{d}_2 \alpha_1 + c_2 d_1 \mathring{\alpha}_2 - c_1 d_1 \mathring{\alpha}_2) \\ & + \mathring{\alpha}_2 b\lambda_2(c_1 d_2 \mathring{\alpha}_2 - c_2 d_2 \mathring{\alpha}_2 + c_2 \mathring{d}_2 \alpha_2 - c_1 \mathring{d}_2 \alpha_2) \Big\} \end{aligned} \tag{87}$$

Similarly, solving (85) for $\Delta\ \mathring{B}_1'$, $\Delta\ A_{-1}'$, and $\Delta\ B_{-1}'$ in terms of A_1' and B_1' with further substitutions from (55) yields the following results:

$$\begin{aligned} \Delta \mathring{B}_1' = \frac{2S}{\mathring{\lambda}_1(\alpha_2^2 - \alpha_1^2)} \Big\{ & a\lambda_1(c_1 d_1 \alpha_2 - c_1 d_2 \alpha_1 + \mathring{c}_1 d_2 \alpha_1 - \mathring{c}_1 d_1 \alpha_2) \\ & + a\lambda_2(c_2 d_2 \alpha_1 - c_2 d_1 \alpha_2 + \mathring{c}_1 d_1 \alpha_2 - \mathring{c}_1 d_2 \alpha_1) \\ & + \mathring{\alpha}_1 b\lambda_1(c_1 d_1 \mathring{\alpha}_1 - c_2 d_1 \mathring{\alpha}_1 + c_2 \mathring{d}_1 \alpha_1 - c_1 \mathring{d}_1 \alpha_1) \\ & + \mathring{\alpha}_1 b\lambda_2(c_2 d_2 \mathring{\alpha}_1 - c_1 d_2 \mathring{\alpha}_1 + c_1 \mathring{d}_1 \alpha_2 - c_2 \mathring{d}_1 \mathring{\alpha}_2) \Big\} \end{aligned} \tag{88}$$

$$\begin{aligned}
\Delta \underline{A}_1' = \frac{2S}{\lambda_1^{\circ}\lambda_2^{\circ}(\alpha_2^2-\alpha_1^2)} \Big\{ & a^2\lambda_1\big[\mathring{c}_1(d_2\alpha_1 - d_1\alpha_2) + \mathring{c}_2(d_1\alpha_2 - d_2\alpha_1)\big] \\
& + a\mathring{\alpha}_1 b\lambda_1\big[\mathring{c}_2\mathring{\alpha}_1(d_1+d_2) - \mathring{c}_2\mathring{d}_1(\alpha_2+\alpha_1) + \mathring{d}_1(c_1\alpha_2 + c_2\alpha_1) - \mathring{\alpha}_1(c_1d_2 + c_2d_1)\big] \\
& + a\mathring{\alpha}_2 b\lambda_1\big[\mathring{c}_1\mathring{d}_2(\alpha_2+\alpha_1) - \mathring{c}_1\mathring{\alpha}_2(d_1+d_2) + \mathring{\alpha}_2(c_1d_2 + c_2d_1) - \mathring{d}_2(c_1\alpha_2 + c_2\alpha_1)\big] \\
& + \mathring{\alpha}_1\mathring{\alpha}_2 b^2\lambda_1\big[\mathring{d}_1\mathring{\alpha}_2(c_1 - c_2) + \mathring{d}_2\mathring{\alpha}_1(c_2 - c_1)\big] \\
& + 2a\mathring{\alpha}_1 b\lambda_2\big[\mathring{c}_2\mathring{d}_1\alpha_2 - \mathring{c}_2\mathring{\alpha}_1 d_2 + c_2 d_2\mathring{\alpha}_1 - c_2\mathring{d}_1\alpha_2\big] \\
& + 2a\mathring{\alpha}_2 b\lambda_2\big[\mathring{c}_1 d_2\mathring{\alpha}_2 - \mathring{c}_1\mathring{d}_2\alpha_2 + c_2\mathring{d}_2\alpha_2 - c_2 d_2\mathring{\alpha}_2\big] \Big\}
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
\Delta \underline{B}_1' = \frac{2S}{\lambda_1^{\circ}\lambda_2^{\circ}(\alpha_2^2-\alpha_1^2)} \Big\{ & 2a\mathring{\alpha}_1 b\lambda_1\big[\mathring{c}_2\mathring{d}_1\alpha_1 - \mathring{c}_2 d_1\mathring{\alpha}_1 - c_1\mathring{d}_1\mathring{\alpha}_1 + c_1 d_1\mathring{\alpha}_1\big] \\
& + 2a\mathring{\alpha}_2 b\lambda_1\big[\mathring{c}_1 d_1\mathring{\alpha}_2 - \mathring{c}_1\mathring{d}_2\alpha_1 + c_1\mathring{d}_2\alpha_1 - c_1 d_1\mathring{\alpha}_2\big] \\
& + a^2\lambda_2\big[(d_1\alpha_2 - d_2\alpha_1)\mathring{c}_1 + (d_2\alpha_1 - d_1\alpha_2)\mathring{c}_2\big] + a\mathring{\alpha}_1 b\lambda_2\big[\mathring{c}_2\mathring{\alpha}_1(d_1+d_2) \\
& + \mathring{d}_1(c_1\alpha_2 + c_2\alpha_1) - \mathring{c}_2\mathring{d}_1(\alpha_1+\alpha_2) - \mathring{\alpha}_1(c_1d_2 + c_2d_1)\big] \\
& + a\mathring{\alpha}_2 b\lambda_2\big[\mathring{c}_1\mathring{d}_2(\alpha_2+\alpha_1) - \mathring{c}_1\mathring{\alpha}_2(d_1+d_2) + \mathring{\alpha}_2(c_1d_2 + c_2d_1) \\
& - \mathring{d}_2(c_1\alpha_2 + c_2\alpha_1)\big] + \mathring{\alpha}_1\mathring{\alpha}_2 b^2\big[(\mathring{d}_2\mathring{\alpha}_1 - \mathring{d}_1\mathring{\alpha}_2)(c_1 - c_2)\big] \Big\}
\end{aligned} \tag{90}$$

Hence from (86), (87), (88), (89) and (90) the following expressions are obtained for the coefficients.

$$
\begin{aligned}
\dot{A}'_1 = \frac{S\,\mathring{\lambda}_1}{2(\alpha_2^2-\alpha_1^2)(den)} \Big\{ & a\Big[\lambda_1\left(c_1 d_2 \alpha_1 - c_1 d_1 \alpha_2 + \mathring{c}_2 d_1 \alpha_2 - \mathring{c}_2 d_2 \alpha_1\right) \\
& + \lambda_2\left(c_2 d_1 \alpha_2 - c_2 d_2 \alpha_1 + \mathring{c}_2 d_2 \alpha_1 - \mathring{c}_2 d_1 \alpha_2\right)\Big] \\
& + \mathring{\alpha}_2 b\Big[\lambda_1\left(c_1 \mathring{d}_2 \alpha_1 - c_2 \mathring{d}_2 \alpha_1 + c_2 d_1 \mathring{\alpha}_2 - c_1 d_1 \mathring{\alpha}_2\right) \\
& + \lambda_2\left(c_1 d_2 \mathring{\alpha}_2 - c_2 d_2 \mathring{\alpha}_2 + c_2 \mathring{d}_2 \alpha_2 - c_1 \mathring{d}_2 \alpha_2\right)\Big]\Big\}
\end{aligned}
\qquad (91)
$$

$$
\begin{aligned}
\dot{B}'_1 = \frac{S\,\mathring{\lambda}_2}{2(\alpha_2^2-\alpha_1^2)(den)} \Big\{ & a\Big[\lambda_1\left(c_1 d_1 \alpha_2 - c_1 d_2 \alpha_1 + \mathring{c}_1 d_2 \alpha_1 - \mathring{c}_1 d_1 \alpha_2\right) \\
& + \lambda_2\left(c_2 d_2 \alpha_1 - c_2 d_1 \alpha_2 + \mathring{c}_1 d_1 \alpha_2 - \mathring{c}_1 d_2 \alpha_1\right)\Big] \\
& + \mathring{\alpha}_1 b\Big[\lambda_1\left(c_1 d_1 \mathring{\alpha}_1 - c_2 d_1 \mathring{\alpha}_1 + c_2 \mathring{d}_1 \alpha_1 - c_1 \mathring{d}_1 \alpha_1\right) \\
& + \lambda_2\left(c_2 d_2 \mathring{\alpha}_1 - c_1 d_2 \mathring{\alpha}_1 + c_1 \mathring{d}_1 \alpha_2 - c_2 \mathring{d}_1 \alpha_2\right)\Big]\Big\}
\end{aligned}
\qquad (92)
$$

$$
\begin{aligned}
A'_{-1} = \frac{S}{2(\alpha_2^2-\alpha_1^2)(den)} \Big\{ & a^2 \lambda_1\Big[(\mathring{c}_1 - \mathring{c}_2)(d_2 \alpha_1 - d_1 \alpha_2)\Big] \\
& + \mathring{\alpha}_1 \mathring{\alpha}_2 b^2 \lambda_1\Big[(c_1 - c_2)(\mathring{d}_1 \mathring{\alpha}_2 - \mathring{d}_2 \mathring{\alpha}_1)\Big] + a\,\mathring{\alpha}_1 b \lambda_1\Big[\mathring{c}_2 \mathring{\alpha}_1 (d_1 + d_2) \\
& - \mathring{c}_2 \mathring{d}_1 (\alpha_1 + \alpha_2) + \mathring{d}_1 (c_1 \alpha_2 + c_2 \alpha_1) - \mathring{\alpha}_1 (c_1 d_2 + c_2 d_1)\Big] \\
& + a\,\mathring{\alpha}_2 b \lambda_1\Big[\mathring{c}_1 \mathring{d}_2 (\alpha_1 + \alpha_2) - \mathring{c}_1 \mathring{\alpha}_2 (d_1 + d_2) + \mathring{\alpha}_2 (c_1 d_2 + c_2 d_1) \\
& - \mathring{d}_2 (c_1 \alpha_2 + c_2 \alpha_1)\Big] + 2a\,\mathring{\alpha}_1 b \lambda_2\Big[\mathring{c}_2 \mathring{d}_1 \alpha_2 - \mathring{c}_2 \mathring{\alpha}_1 d_2 + \mathring{\alpha}_1 c_2 d_2 \\
& - \mathring{d}_1 c_2 \alpha_2\Big] + 2a\,\mathring{\alpha}_2 b\, \lambda_2\Big[\mathring{c}_1 \mathring{\alpha}_2 d_2 - \mathring{c}_1 \mathring{d}_2 \alpha_2 \\
& + \mathring{d}_2 c_2 \alpha_2 - \mathring{\alpha}_2 c_2 d_2\Big]\Big\}
\end{aligned}
\qquad (93)
$$

and

$$\underline{B}_1' = \frac{S}{2(\alpha_2^2-\alpha_1^2)(\mathrm{den})}\Big\{a^2\lambda_2\Big[(\mathring{c}_1-\mathring{c}_2)(d_1\alpha_2-d_2\alpha_1)\Big]$$

$$+\mathring{\alpha}_1\mathring{\alpha}_2 b^2\lambda_2\Big[(c_1-c_2)(\mathring{d}_2\mathring{\alpha}_1-\mathring{d}_1\mathring{\alpha}_2)\Big]+a\mathring{\alpha}_1 b\lambda_2\Big[\mathring{c}_2\mathring{\alpha}_1(d_1+d_2)$$

$$-\mathring{c}_2\mathring{d}_1(\alpha_1+\alpha_2)+\mathring{d}_1(c_1\alpha_2+c_2\alpha_1)-\mathring{\alpha}_1(c_1d_2+c_2d_1)\Big]$$

$$+a\mathring{\alpha}_2 b\lambda_2\Big[\mathring{c}_1\mathring{d}_2(\alpha_1+\alpha_2)-\mathring{c}_1\mathring{\alpha}_2(d_1+d_2)-\mathring{d}_2(c_1\alpha_2+c_2\alpha_1)$$

$$+\mathring{\alpha}_2(c_1d_2+c_2d_1)\Big]+2a\mathring{\alpha}_1 b\lambda_1\Big[\mathring{c}_2\mathring{d}_1\alpha_1-\mathring{c}_2\mathring{\alpha}_1 d_1-\mathring{d}_1 c_1\alpha_1$$

$$+\mathring{\alpha}_1 c_1 d_1\Big]+2a\mathring{\alpha}_2 b\lambda_1\Big[\mathring{c}_1\mathring{\alpha}_2 d_1-\mathring{c}_1\mathring{d}_2\alpha_1$$

$$+\mathring{d}_2 c_1\alpha_1-\mathring{\alpha}_2 c_1 d_1\Big]\Big\} \qquad (94)$$

where

$$(\mathrm{den}) = \Big\{a^2\Big[(\mathring{c}_1-\mathring{c}_2)(d_2\alpha_1-d_1\alpha_2)\Big]+a\mathring{\alpha}_1 b\Big[\mathring{c}_2\mathring{d}_1(\alpha_2-\alpha_1)$$

$$+\mathring{c}_2\mathring{\alpha}_1(d_1-d_2)+\mathring{d}_1(c_2\alpha_1-c_1\alpha_2)+\mathring{\alpha}_1(c_1d_2-c_2d_1)\Big]$$

$$+a\mathring{\alpha}_2 b\Big[\mathring{c}_1\mathring{d}_2(\alpha_1-\alpha_2)-\mathring{c}_1\mathring{\alpha}_2(d_1-d_2)+\mathring{d}_2(c_1\alpha_2-c_2\alpha_1)$$

$$+\mathring{\alpha}_2(c_2d_1-c_1d_2)\Big]+\mathring{\alpha}_1\mathring{\alpha}_2 b^2\Big[(c_1-c_2)(\mathring{d}_2\mathring{\alpha}_1-\mathring{d}_1\mathring{\alpha}_2)\Big]\Big\}. \qquad (95)$$

Since all imaginary coefficients have been discarded, the expressions for the plate stresses now become (as seen from equations 39, 37 and 40)

$$\begin{aligned}
\sigma_x &= 2R\left\{\frac{-\alpha_1^2}{\lambda_1\zeta_1^2-\mu_1}\left[A_1'\zeta_1^2-A_{-1}'\right]-\frac{\alpha_2^2}{\lambda_2\zeta_2^2-\mu_2}\left[B_1'\zeta_2^2-B_{-1}'\right]\right\}\\
\sigma_y &= 2R\left\{\frac{1}{\lambda_1\zeta_1^2-\mu_1}\left[A_1'\zeta_1^2-A_{-1}'\right]+\frac{1}{\lambda_2\zeta_2^2-\mu_2}\left[B_1'\zeta_2^2-B_{-1}'\right]\right\}\\
\tau_{xy} &= R\left\{\frac{-i\alpha_1}{\lambda_1\zeta_1^2-\mu_1}\left[A_1'\zeta_1^2-A_{-1}'\right]-\frac{i\alpha_2}{\lambda_2\zeta_2^2-\mu_2}\left[B_1'\zeta_2^2-B_{-1}'\right]\right\}
\end{aligned}\tag{96}$$

and from (32) the disk stresses reduce to

$$\begin{aligned}
\mathring{\sigma}_x &= 2R\left\{\frac{-\mathring{\alpha}_1^2}{\mathring{\lambda}_1\mathring{\zeta}_1^2-\mathring{\mu}_1}\left[\mathring{A}_1'\mathring{\zeta}_1^2-\mathring{A}_{-1}'\right]-\frac{\mathring{\alpha}_2^2}{\mathring{\lambda}_2\mathring{\zeta}_2^2-\mathring{\mu}_2}\left[\mathring{B}_1'\mathring{\zeta}_2^2-\mathring{B}_{-1}'\right]\right\}\\
\mathring{\sigma}_y &= 2R\left\{\frac{1}{\mathring{\lambda}_1\mathring{\zeta}_1^2-\mathring{\mu}_1}\left[\mathring{A}_1'\mathring{\zeta}_1^2-\mathring{A}_{-1}'\right]+\frac{1}{\mathring{\lambda}_2\mathring{\zeta}_2^2-\mathring{\mu}_2}\left[\mathring{B}_1'\mathring{\zeta}_2^2-\mathring{B}_{-1}'\right]\right\}\\
\mathring{\tau}_{xy} &= R\left\{\frac{-i\mathring{\alpha}_1}{\mathring{\lambda}_1\mathring{\zeta}_1^2-\mathring{\mu}_1}\left[\mathring{A}_1'\mathring{\zeta}_1^2-\mathring{A}_{-1}'\right]-\frac{i\mathring{\alpha}_2}{\mathring{\lambda}_2\mathring{\zeta}_2^2-\mathring{\mu}_2}\left[\mathring{B}_1'\mathring{\zeta}_2^2-\mathring{B}_{-1}'\right]\right\}
\end{aligned}\tag{97}$$

where ζ_1 and ζ_2 are given by (29) and $\mathring{\zeta}_1$ and $\mathring{\zeta}_2$ are similar expressions for the disk.

Now we have

$$\mathring{A}_{-1}' = \mathring{A}_1'\left(\frac{\mathring{\mu}_1}{\mathring{\lambda}_1}\right) \quad , \quad \mathring{B}_{-1}' = \mathring{B}_1'\left(\frac{\mathring{\mu}_2}{\mathring{\lambda}_2}\right)$$

by equations (33) and (34); and if W_1 and W_2 are defined as

$$W_1 = \sqrt{z_1^2 - \lambda_1\mu_1} \qquad \text{and} \qquad W_2 = \sqrt{z_2^2 - \lambda_2\mu_2}$$

then with these substitutions the plate and disk stresses reduce to

$$\sigma_x = R\left\{\frac{-\alpha_1^2}{W_1(z_1+W_1)}\left[\frac{(z_1+W_1)^2}{\lambda_1}A_1' - \lambda_1 A_{-1}'\right] - \frac{\alpha_2^2}{W_2(z_2+W_2)}\left[\frac{(z_2+W_2)^2}{\lambda_2}B_1' - \lambda_2 B_{-1}'\right]\right\}$$

$$\sigma_y = R\left\{\frac{1}{W_1(z_1+W_1)}\left[\frac{(z_1+W_1)^2}{\lambda_1}A_1' - \lambda_1 A_{-1}'\right] + \frac{1}{W_2(z_2+W_2)}\left[\frac{(z_2+W_2)^2}{\lambda_2}B_1' - \lambda_2 B_{-1}'\right]\right\} \tag{98}$$

$$\tau_{xy} = \frac{R}{2}\left\{\frac{-i\alpha_1}{W_1(z_1+W_1)}\left[\frac{(z_1+W_1)^2}{\lambda_1}A_1' - \lambda_1 A_{-1}'\right] - \frac{i\alpha_2}{W_2(z_2+W_2)}\left[\frac{(z_2+W_2)^2}{\lambda_2}B_1' - \lambda_2 B_{-1}'\right]\right\}$$

$$\mathring{\sigma}_x = 2R\left\{\frac{-\mathring{\alpha}_1^2}{\mathring{\lambda}_1}\mathring{A}_1' - \frac{\mathring{\alpha}_2^2}{\mathring{\lambda}_2}\mathring{B}_1'\right\}$$

$$\mathring{\sigma}_y = 2R\left\{\frac{\mathring{A}_1'}{\mathring{\lambda}_1} + \frac{\mathring{B}_1'}{\mathring{\lambda}_2}\right\} \tag{99}$$

$$\mathring{\tau}_{xy} = R\left\{\frac{-i\mathring{\alpha}_1}{\mathring{\lambda}_1}\mathring{A}_1' - \frac{i\mathring{\alpha}_2}{\mathring{\lambda}_2}\mathring{B}_1'\right\}$$

where $\mathring{A}_1'$, $\mathring{B}_1'$, A_1', B_1', A_{-1}' and B_{-1}' are defined by (91), (92), (55), (55), (93) and (94) respectively.

Equations (98) and (99) represent the complete solution of the plane stresses for the problem of an elliptical disk of orthotropic material imbedded in an infinite rectangular orthotropic plate which is subjected to a tension (or compression) S on two opposing edges.

It will be advantageous in the following sections to have the disk stresses (99) rewritten with the substitutions (91) and (92). These results follow:

$$\begin{aligned}\mathring{\sigma}_x = \frac{S}{(\alpha_2^2-\alpha_1^2)(den)}\Big[&-\mathring{\alpha}_1^2 a\left(\lambda_1 c_1 d_2 \alpha_1 - \lambda_1 c_1 d_1 \alpha_2 + \lambda_2 c_2 d_1 \alpha_2 - \lambda_2 c_2 d_2 \alpha_1\right) \\ &-\mathring{\alpha}_1^2 \mathring{c}_2 a\left(\lambda_1 d_1 \alpha_2 - \lambda_1 d_2 \alpha_1 + \lambda_2 d_2 \alpha_1 - \lambda_2 d_1 \alpha_2\right) \\ &-\mathring{\alpha}_1^2 \mathring{\alpha}_2 \mathring{d}_2 b\left(\lambda_1 c_1 \alpha_1 - \lambda_1 c_2 \alpha_1 + \lambda_2 c_2 \alpha_2 - \lambda_2 c_1 \alpha_2\right) \\ &-\mathring{\alpha}_2^2 a\left(\lambda_1 c_1 d_1 \alpha_2 - \lambda_1 c_1 d_2 \alpha_1 + \lambda_2 c_2 d_2 \alpha_1 - \lambda_2 c_2 d_1 \alpha_2\right) \\ &-\mathring{\alpha}_2^2 \mathring{c}_1 a\left(\lambda_1 d_2 \alpha_1 - \lambda_1 d_1 \alpha_2 + \lambda_2 d_1 \alpha_2 - \lambda_2 d_2 \alpha_1\right) \\ &-\mathring{\alpha}_2^2 \mathring{\alpha}_1 \mathring{d}_1 b\left(\lambda_1 c_2 \alpha_1 - \lambda_1 c_1 \alpha_1 + \lambda_2 c_1 \alpha_2 - \lambda_2 c_2 \alpha_2\right)\Big]\end{aligned} \qquad (100)$$

$$\begin{aligned}\mathring{\sigma}_y = \frac{S}{(\alpha_2^2-\alpha_1^2)(den)}\Big[&-\mathring{c}_2 a\left(\lambda_1 d_2 \alpha_1 - \lambda_1 d_1 \alpha_2 + \lambda_2 d_1 \alpha_2 - \lambda_2 d_2 \alpha_1\right) \\ &-\mathring{c}_1 a\left(\lambda_1 d_1 \alpha_2 - \lambda_1 d_2 \alpha_1 + \lambda_2 d_2 \alpha_1 - \lambda_2 d_1 \alpha_2\right) \\ &-\mathring{\alpha}_2 \mathring{d}_2 b\left(\lambda_1 c_2 \alpha_1 - \lambda_1 c_1 \alpha_1 + \lambda_2 c_1 \alpha_2 - \lambda_2 c_2 \alpha_2\right) \\ &-\mathring{\alpha}_1 \mathring{d}_1 b\left(\lambda_1 c_1 \alpha_1 - \lambda_1 c_2 \alpha_1 + \lambda_2 c_2 \alpha_2 - \lambda_2 c_1 \alpha_2\right) \\ &+\mathring{\alpha}_2^2 b\left(\lambda_1 c_2 d_1 - \lambda_1 c_1 d_1 + \lambda_2 c_1 d_2 - \lambda_2 c_2 d_2\right) \\ &+\mathring{\alpha}_1^2 b\left(\lambda_1 c_1 d_1 - \lambda_1 c_2 d_1 + \lambda_2 c_2 d_2 - \lambda_2 c_1 d_2\right)\Big]\end{aligned} \qquad (101)$$

$$\mathring{\tau}_{xy} = 0$$

It is noted that the shear stress becomes zero since the quantities $\mathring{\alpha}_1$, $\mathring{\alpha}_2$, $\mathring{A}_1'$ and $\mathring{B}_1'$ are all real.

CHAPTER III

DETERMINATION OF STRESSES FOR SPECIAL CASES

Circular Orthotropic Disk in an Orthotropic Plate

It will now be of interest to reduce this solution to several special cases. Consider first a circular disk of orthotropic material instead of the elliptic disk. In this case it is merely necessary to set $b = a$, which results in slight simplification of the preceding expressions. In particular

$$A_1' = \frac{Sa(1+\alpha_1)}{2(\alpha_2^2 - \alpha_1^2)} \quad , \qquad B_1' = \frac{-Sa(1+\alpha_2)}{2(\alpha_2^2 - \alpha_1^2)} \quad , \qquad (102)$$

$$\begin{aligned} A_{-1}' = \frac{Sa}{2(\alpha_2^2 - \alpha_1^2)(den)'} \Big\{ & (1+\alpha_1)\Big[(\mathring{c}_1 - \mathring{c}_2)(d_2\alpha_1 - d_1\alpha_2) \\ & + \mathring{\alpha}_1\mathring{\alpha}_2(c_1 - c_2)(\mathring{d}_1\mathring{\alpha}_2 - \mathring{d}_2\mathring{\alpha}_1) + (d_1 + d_2)(\mathring{c}_2\mathring{\alpha}_1^2 - \mathring{c}_1\mathring{\alpha}_2^2) \\ & + (\alpha_1 + \alpha_2)(\mathring{c}_1\mathring{d}_2\mathring{\alpha}_2 - \mathring{c}_2\mathring{d}_1\mathring{\alpha}_1) + (c_1\alpha_2 + c_2\alpha_1)(\mathring{d}_1\mathring{\alpha}_1 - \mathring{d}_2\mathring{\alpha}_2) \\ & + (\mathring{\alpha}_2^2 - \mathring{\alpha}_1^2)(c_1 d_2 + c_2 d_1)\Big] + 2(1+\alpha_2)\Big[c_2\alpha_2(\mathring{d}_2\mathring{\alpha}_2 - \mathring{d}_1\mathring{\alpha}_1) \\ & - c_2 d_2(\mathring{\alpha}_2^2 - \mathring{\alpha}_1^2) + d_2(\mathring{c}_1\mathring{\alpha}_2^2 - \mathring{c}_2\mathring{\alpha}_1^2) \\ & + \alpha_2(\mathring{\alpha}_1\mathring{c}_2\mathring{d}_1 - \mathring{\alpha}_2\mathring{c}_1\mathring{d}_2)\Big] \Big\} \qquad (103) \end{aligned}$$

and

$$\begin{aligned} B'_{-1} = \frac{Sa}{2(\alpha_2^2-\alpha_1^2)(den)'} \Big\{ & (1+\alpha_2)\Big[(\mathring{c}_1-\mathring{c}_2)(d_1\alpha_2-d_2\alpha_1) \\ & + \mathring{\alpha}_1\mathring{\alpha}_2(c_1-c_2)(\mathring{d}_2\mathring{\alpha}_1-\mathring{d}_1\mathring{\alpha}_2)+(d_1+d_2)(\mathring{c}_2\mathring{\alpha}_1^2-\mathring{c}_1\mathring{\alpha}_2^2) \\ & +(\alpha_1+\alpha_2)(\mathring{c}_1\mathring{d}_2\mathring{\alpha}_2-\mathring{c}_2\mathring{d}_1\mathring{\alpha}_1)+(c_1\alpha_2+c_2\alpha_1)(\mathring{d}_1\mathring{\alpha}_1-\mathring{d}_2\mathring{\alpha}_2) \\ & +(\mathring{\alpha}_2^2-\mathring{\alpha}_1^2)(c_1d_2+c_2d_1)\Big]+2(1+\alpha_1)\Big[c_1\alpha_1(\mathring{d}_2\mathring{\alpha}_2-\mathring{d}_1\mathring{\alpha}_1) \\ & -c_1d_1(\mathring{\alpha}_2^2-\mathring{\alpha}_1^2)+d_1(\mathring{c}_1\mathring{\alpha}_2^2-\mathring{c}_2\mathring{\alpha}_1^2) \\ & +\alpha_1(\mathring{\alpha}_1\mathring{c}_2\mathring{d}_1-\mathring{\alpha}_2\mathring{c}_1\mathring{d}_2)\Big]\Big\} \end{aligned} \tag{104}$$

where

$$\begin{aligned} (den)' = \Big\{ & \Big[(\mathring{c}_1-\mathring{c}_2)(d_2\alpha_1-d_1\alpha_2)\Big] \\ & +\mathring{\alpha}_1\Big[\mathring{c}_2\mathring{d}_1(\alpha_2-\alpha_1)+\mathring{c}_2\mathring{\alpha}_1(d_1-d_2)+\mathring{d}_1(c_2\alpha_1-c_1\alpha_2) \\ & +\mathring{\alpha}_1(c_1d_2-c_2d_1)\Big]+\mathring{\alpha}_2\Big[\mathring{c}_1\mathring{d}_2(\alpha_1-\alpha_2) \\ & -\mathring{c}_1\mathring{\alpha}_2(d_1-d_2)+\mathring{d}_2(c_1\alpha_2-c_2\alpha_1)+\mathring{\alpha}_2(c_2d_1-c_1d_2)\Big] \\ & +\mathring{\alpha}_1\mathring{\alpha}_2\Big[(c_1-c_2)(\mathring{d}_2\mathring{\alpha}_1-\mathring{d}_1\mathring{\alpha}_2)\Big]\Big\} \end{aligned} \tag{105}$$

Equations (102), (103), (104) and (105) are now substituted into the stresses (98) which are further modified by setting

$$\begin{aligned} \lambda_1 &= a(1+\alpha_1) & \mu_1 &= a(1-\alpha_1) \\ \lambda_2 &= a(1+\alpha_2) & \mu_2 &= a(1-\alpha_2) \end{aligned} \tag{106}$$

and

$$W_1 \longrightarrow W_1' = \sqrt{z_1^2 - a^2(1 - \alpha_1^2)}$$
$$W_2 \longrightarrow W_2' = \sqrt{z_2^2 - a^2(1 - \alpha_2^2)} \quad . \tag{107}$$

Stresses for the circular orthotropic disk reduce to

$$\begin{aligned}\mathring{\sigma}_x = \frac{S}{(\alpha_2^2-\alpha_1^2)(den)'}\Big\{&(\mathring{\alpha}_2^2-\mathring{\alpha}_1^2)\Big[(1+\alpha_1)(c_1d_2\alpha_1-c_1d_1\alpha_2)\\ &+(1+\alpha_2)(c_2d_1\alpha_2-c_2d_2\alpha_1)\Big]+(\mathring{\alpha}_2^2\mathring{c}_1-\mathring{\alpha}_1^2\mathring{c}_2)\Big[(1+\alpha_1)(d_1\alpha_2-d_2\alpha_1)\\ &+(1+\alpha_2)(d_2\alpha_1-d_1\alpha_2)\Big]+(\mathring{\alpha}_2^2\mathring{\alpha}_1\mathring{d}_1-\mathring{\alpha}_1^2\mathring{\alpha}_2\mathring{d}_2)\Big[(1+\alpha_1)(c_1\alpha_1-c_2\alpha_1)\\ &+(1+\alpha_2)(c_2\alpha_2-c_1\alpha_2)\Big]\Big\}\end{aligned} \tag{108}$$

$$\begin{aligned}\mathring{\sigma}_y = \frac{S}{(\alpha_2^2-\alpha_1^2)(den)'}\Big\{&(\mathring{\alpha}_2^2-\mathring{\alpha}_1^2)\Big[(1+\alpha_1)(c_2d_1-c_1d_1)+(1+\alpha_2)(c_1d_2-c_2d_1)\\ &+(\mathring{\alpha}_2\mathring{d}_2-\mathring{\alpha}_1\mathring{d}_1)\Big[(1+\alpha_1)(c_1\alpha_1-c_2\alpha_1)+(1+\alpha_2)(c_2\alpha_2-c_1\alpha_2)\Big]\\ &+(\mathring{c}_2-\mathring{c}_1)\Big[(1+\alpha_1)(d_1\alpha_2-d_2\alpha_1)+(1+\alpha_2)(d_2\alpha_1-d_1\alpha_2)\Big]\Big\}\end{aligned} \tag{109}$$

$$\mathring{\tau}_{xy} = 0 \quad .$$

Isotropic Elliptic Disk in an Orthotropic Plate

For the case in which the disk material becomes isotropic, it is necessary that

$$\mathring{E}_x = \mathring{E}_y = \mathring{E} \quad \text{and} \quad \mathring{\nu}_{xy} = \mathring{\nu}_{yx} = \mathring{\nu}$$

since the elastic properties are independent of direction. Hence $\mathring{\epsilon} = 1$ so that $\mathring{\alpha}_1 = \mathring{\alpha}$ and $\mathring{\alpha}_2 = \mathring{\beta}$.

Now $\mathring{\alpha}_2^{\,2}$ and $\dot{\beta}^2$ as used by Professor Smith[11] have the values e^{ϕ} and $e^{-\phi}$ respectively. The function

$$\dot{\Phi} = \mathring{F}_1(x + i\mathring{\alpha}_1 y) + \mathring{F}_2(x + i\mathring{\alpha}_2 y)$$

was obtained from the compatability equation

$$\left[\frac{\partial^2}{\partial x^2} + \alpha^2 \frac{\partial^2}{\partial \eta^2}\right]\left[\frac{\partial^2}{\partial x^2} + \dot{\beta}^2 \frac{\partial^2}{\partial \eta^2}\right]\mathring{F} = 0$$

where $\eta = \mathring{\epsilon}\,y$ and $i e^{\phi/2}$, $i e^{-\phi/2}$; $-i e^{\phi/2}$; $-i e^{-\phi/2}$ are roots of the auxiliary quadratic equation

$$\delta^4 + 2\mathring{K}\delta^2 + 1 = 0$$

in which

$$\mathring{K} = \cosh\phi = \frac{\sqrt{\mathring{E}_x\,\mathring{E}_y}}{2}\left(\frac{1}{\mathring{\mu}_{xy}} - \frac{2\mathring{\nu}_{xy}}{\dot{E}_x}\right)$$

and $\delta^2 = -e^{\phi}$ or $-e^{-\phi}$.

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[11] *Effect of Elliptic or Circular Holes . . . .*, pp. 4-5.

Consequently the following relationships exist for the isotropic elliptic disk in which ϕ tends to zero.

$$\mathring{\alpha}_1^2 = e^{\phi} \qquad \mathring{\alpha}_2^2 = e^{-\phi}$$

$$\mathring{\alpha}_1 = e^{\phi/2} \qquad \mathring{\alpha}_2 = e^{-\phi/2}$$

$$\mathring{\lambda}_1 = a + b\,e^{\phi/2} \qquad \mathring{\lambda}_2 = a + b\,e^{-\phi/2}$$

$$\mathring{\mu}_1 = a - b\,e^{\phi/2} \qquad \mathring{\mu}_2 = a - b\,e^{-\phi/2}$$

$$\mathring{z}_1 = x + iy\,e^{\phi/2} \qquad \mathring{z}_2 = x + iy\,e^{-\phi/2}$$

$$\mathring{E}_x = \mathring{E}_y = \mathring{E} \qquad \mathring{\epsilon} = 1$$

$$\mathring{\nu}_{xy} = \mathring{\nu}_{yx} = \mathring{\nu}$$

$$\mathring{s}_{11} = \frac{1}{\mathring{E}} \quad , \quad \mathring{s}_{22} = \frac{1}{\mathring{E}} \quad , \quad \mathring{s}_{12} = \frac{-\mathring{\nu}}{\mathring{E}}$$

$$\mathring{s}_{33} = \frac{1}{\mathring{\mu}} = \frac{2(1+\mathring{\nu})}{\mathring{E}}$$

$$\mathring{c}_1 = \frac{-1}{\mathring{E}}\left(e^{\phi} + \mathring{\nu}\right) \quad , \quad \mathring{d}_1 = \frac{1}{\mathring{E}}\left(\frac{e^{\phi}\mathring{\nu} + 1}{e^{\phi/2}}\right)$$

$$\mathring{c}_2 = \frac{-1}{\mathring{E}}\left(e^{-\phi} + \mathring{\nu}\right) \quad , \quad \mathring{d}_2 = \frac{1}{\mathring{E}}\left(\frac{e^{-\phi}\mathring{\nu} + 1}{e^{-\phi/2}}\right)$$

For the relationships above $\mathring{\sigma}_x$, $\mathring{\sigma}_y$, A'_{-1} , and B'_{-1} take the indeterminant form 0/0 . Evaluating by differentiating both numerator and denominator with respect to ϕ , and then letting $\phi = 0$ and $\mathring{\epsilon} = 1$, it follows that

$$\mathring{\sigma}_x = \frac{S\mathring{E}}{(\alpha_2^2 - \alpha_1^2)\,[DEN]}\Big\{a\mathring{E}\big[(\lambda_1 c_1 - \lambda_2 c_2)(d_1\alpha_2 - d_2\alpha_1)\big]$$

$$+ b\big[(\lambda_1\alpha_1 - \lambda_2\alpha_2)(c_2 - c_1)\big] + \mathring{\nu} a\big[(\lambda_1 - \lambda_2)(d_1\alpha_2 - d_2\alpha_1)\big]\Big\} \qquad (110)$$

$$\mathring{\sigma}_y = \frac{S\mathring{E}}{(\alpha_2^2 - \alpha_1^2)\,[DEN]}\Big\{b\mathring{E}\big[(\lambda_1 d_1 - \lambda_2 d_2)(c_1 - c_2)\big]$$

$$+ a\big[(\lambda_1 - \lambda_2)(d_1\alpha_2 - d_2\alpha_1)\big] + \mathring{\nu} b\big[\lambda_2(c_1\alpha_2 - c_2\alpha_2) + \lambda_1(c_2\alpha_1 - c_1\alpha_1)\big]\Big\} \qquad (111)$$

$$\mathring{\tau}_{xy} = 0$$

where

$$[\mathrm{DEN}] = \big[a^2\mathring{E}(d_1\alpha_2 - d_2\alpha_1) + ab\mathring{\nu}\mathring{E}(d_2 - d_1 + c_2\alpha_1 - c_1\alpha_2)$$

$$+ b^2\mathring{E}(c_1 - c_2) + ab\mathring{E}^2(c_1 d_2 - c_2 d_1) + ab(1 - \mathring{\nu}^2)(\alpha_2 - \alpha_1)\big] \qquad (112)$$

and

$$\underline{A}'_{-1} = \frac{S}{2(\alpha_2^2-\alpha_1^2)[DEN]}\Big\{ \mathring{E}^2\Big[2ab\lambda_2 c_2 d_2 - ab\lambda_1(c_1 d_2 + c_2 d_1)\Big]$$
$$+ \mathring{E}\Big[a^2\lambda_1(d_1\alpha_2 - d_2\alpha_1) + ab\lambda_1\mathring{\nu}\left(c_1\alpha_2 + c_2\alpha_1 - d_1 - d_2\right)$$
$$+ b^2\lambda_1(c_2 - c_1) + 2ab\lambda_2\mathring{\nu}(d_2 - c_2\alpha_2)\Big]$$
$$+ ab(1-\mathring{\nu}^2)\Big[2\lambda_2\alpha_2 - \lambda_1(\alpha_1+\alpha_2)\Big]\Big\} \tag{113}$$

$$\underline{B}'_{-1} = \frac{S}{2(\alpha_2^2-\alpha_1^2)[DEN]}\Big\{ \mathring{E}^2\Big[2ab\lambda_1 c_1 d_1 - ab\lambda_2(c_1 d_2 + c_2 d_1)\Big]$$
$$+ \mathring{E}\Big[a^2\lambda_2(d_2\alpha_1 - d_1\alpha_2) + ab\lambda_2\mathring{\nu}\left(c_1\alpha_2 + c_2\alpha_1 - d_1 - d_2\right)$$
$$+ b^2\lambda_2(c_1 - c_2) + 2ab\lambda_1\mathring{\nu}(d_1 - c_1\alpha_1)\Big]$$
$$+ ab(1-\mathring{\nu}^2)\Big[2\lambda_1\alpha_1 - \lambda_2(\alpha_1+\alpha_2)\Big]\Big\} . \tag{114}$$

Expressions (55), (113) and (114) substituted into (98) plus the stresses (110) and (111) now give the complete solution for the case of an elliptic isotropic disk in an orthotropic plate.

It will be of further interest now to reduce this solution to the case of the circular isotropic disk in an infinite orthotropic plate.

Circular Disk of Isotropic Material in an Orthotropic Plate

For the special case of a circular isotropic disk in an orthotropic plate we follow a previous procedure by setting $b = a$, where it should be noted that both λ_1 and λ_2 contain a and b.

Rewriting $\mathring{\sigma}_x$ and $\mathring{\sigma}_y$ from (110) and (111) with the substitutions $\lambda_1 = a + \alpha_1 b$ and $\lambda_2 = a + \alpha_2 b$ and then letting $b \rightarrow a$ yields the following results:

$$\mathring{\sigma}_x = \frac{S\,\mathring{E}}{(\alpha_2^2-\alpha_1^2)\,\textcircled{\text{DEN}}}\left\{\mathring{E}(d_1\alpha_2-d_2\alpha_1)\left[c_1(1+\alpha_1)-c_2(1+\alpha_2)\right]\right.$$

$$\left.+(c_1-c_2)(\alpha_2-\alpha_1)\left[\alpha_2+\alpha_1+1\right]+\mathring{\nu}(\alpha_1-\alpha_2)(d_1\alpha_2-d_2\alpha_1)\right\} \qquad (115)$$

$$\mathring{\sigma}_y = \frac{S\,\mathring{E}}{(\alpha_2^2-\alpha_1^2)\,\textcircled{\text{DEN}}}\left\{\mathring{E}(c_1-c_2)\left[d_1(1+\alpha_1)-d_2(1+\alpha_2)\right]\right.$$

$$+\left[(\alpha_1-\alpha_2)(d_1\alpha_2-d_2\alpha_1)\right]$$

$$\left.+\mathring{\nu}\left[(1+\alpha_2)(c_1\alpha_2-c_2\alpha_2)+(1+\alpha_1)(c_2\alpha_1-c_1\alpha_1)\right]\right\} \qquad (116)$$

$$\mathring{\tau}_{xy} = 0$$

where

$$\textcircled{\text{DEN}} = \left[\mathring{E}^2(c_1d_2-c_2d_1)+\mathring{E}(d_1\alpha_2-d_2\alpha_1+c_1-c_2)\right.$$

$$\left.+\mathring{\nu}\,\mathring{E}(c_2\alpha_1-c_1\alpha_2+d_2-d_1)+(1-\mathring{\nu}^2)(\alpha_2-\alpha_1)\right]\;. \qquad (117)$$

Also

$$A'_{-1} = \frac{S a}{2(\alpha_2^2 - \alpha_1^2)\enclose{circle}{\mathrm{DEN}}} \Big\{ \mathring{E}^2 \Big[2c_2 d_2 (1+\alpha_2) - (1+\alpha_1)(c_1 d_2 + c_2 d_1) \Big]$$

$$+ \mathring{E} \Big[(1+\alpha_1)(d_1\alpha_2 - d_2\alpha_1 + c_2 - c_1) + \mathring{\nu}(1+\alpha_1)(c_1\alpha_2 + c_2\alpha_1 - d_1 - d_2)$$

$$+ 2\mathring{\nu}(1+\alpha_2)(d_2 - c_2\alpha_2) \Big] + (1-\mathring{\nu}^2) \Big[2\alpha_2(1+\alpha_2) - (1+\alpha_1)(\alpha_1 + \alpha_2) \Big] \Big\} \tag{118}$$

and

$$B'_{-1} = \frac{S a}{2(\alpha_2^2 - \alpha_1^2)\enclose{circle}{\mathrm{DEN}}} \Big\{ \mathring{E}^2 \Big[2c_1 d_1 (1+\alpha_1) - (1+\alpha_2)(c_1 d_2 + c_2 d_1) \Big]$$

$$+ \mathring{E} \Big[(1+\alpha_2)(d_2\alpha_1 - d_1\alpha_2 + c_1 - c_2) + \mathring{\nu}(1+\alpha_2)(c_1\alpha_2 + c_2\alpha_1 - d_1 - d_2)$$

$$+ 2\mathring{\nu}(1+\alpha_1)(d_1 - c_1\alpha_1) \Big] + (1-\mathring{\nu}^2) \Big[2\alpha_1(1+\alpha_1) - (1+\alpha_2)(\alpha_1 + \alpha_2) \Big] \Big\} . \tag{119}$$

For the circular isotropic disk in an orthotropic plate, the constants A'_1 and B'_1 take the values (102), and equations (98) are modified by the relationships (106) and (107).

Elliptic Hole in an Orthotropic Plate

If the orthotropic plate contains an elliptic hole instead of a disk, then all the disk constants reduce to zero, viz.,

$$\overset{\circ}{E} = \overset{\circ}{\nu} = 0$$

This substitution makes $\overset{\circ}{\sigma}_x = \overset{\circ}{\sigma}_y = \overset{\circ}{\tau}_{xy} = 0$ as seen from equations (110), (111) and (112). Then from (55), (112), (113) and (114) the plate constants are found to reduce to

$$A'_1 = \frac{S(a+\alpha_1 b)}{2(\alpha_2^2-\alpha_1^2)} \quad , \qquad B'_1 = \frac{-S(a+\alpha_2 b)}{2(\alpha_2^2-\alpha_1^2)}$$

$$A'_{-1} = \frac{S}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{2\alpha_2(a+\alpha_2 b)-(\alpha_1+\alpha_2)(a+\alpha_1 b)}{\alpha_2-\alpha_1}\right] \tag{120}$$

$$B'_{-1} = \frac{S}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{2\alpha_1(a+\alpha_1 b)-(\alpha_1+\alpha_2)(a+\alpha_2 b)}{\alpha_2-\alpha_1}\right] .$$

These constants substituted into (98) now provide the stresses for this particular situation.

A comparison of these results with those obtained by Smith[12] may not at once appear identical. However a slightly different mode of attack may clarify the situation.

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[12] Ibid., p. 9.

By the results of Chapter II, equation (35) becomes

$$\frac{\partial \Phi}{\partial x} = R\left\{ A_1' \rho_1 + \frac{A_1'}{\rho_1} + B_1' \rho_2 + \frac{B_1'}{\rho_2} \right\}$$

Substituting values from (120) now yields

$$\frac{\partial \Phi}{\partial x} = R\left\{ \frac{S}{2(\alpha_2^2-\alpha_1^2)} \left[(a+\alpha_1 b)\rho_1 + \frac{(a-\alpha_1 b)}{\rho_1} + \frac{2b(\alpha_2-\alpha_1)}{\rho_1} \right.\right.$$

$$\left.\left. - (a+\alpha_2 b)\rho_2 - \frac{(a-\alpha_2 b)}{\rho_2} - \frac{2b(\alpha_2+\alpha_1)}{\rho_2} \right] \right\} . \qquad (121)$$

Now since

$$2z_1 = (a+\alpha_1 b)\rho_1 + \frac{(a - \alpha_1 b)}{\rho_1}$$

$$2z_2 = (a+\alpha_2 b)\rho_2 + \frac{(a - \alpha_2 b)}{\rho_2}$$

and

$$\rho_1 = \frac{z_1 + w_1}{a+\alpha_1 b} \quad , \quad \rho_2 = \frac{z_2 + w_2}{a+\alpha_2 b}$$

then equation (121) further reduces to

$$\frac{\partial \Phi}{\partial x} = R\left\{ \frac{S b}{(\alpha_2-\alpha_1)} \left[\frac{a+\alpha_1 b}{z_1+w_1} - \frac{a+\alpha_2 b}{z_2+w_2} \right] \right\} . \qquad (122)$$

Similarly,

$$\frac{\partial \Phi}{\partial y} = R\left\{ i\alpha_1\left[A_1' \zeta_1 + \frac{A_{-1}'}{\zeta_1} \right] + i\alpha_2\left[B_1' \zeta_2 + \frac{B_{-1}'}{\zeta_2} \right] \right\}$$

$$= R\left\{ \frac{Sb}{(\alpha_2-\alpha_1)} \left[\frac{i\alpha_1}{\zeta_1} - \frac{i\alpha_2}{\zeta_2} \right] \right\} + Sy$$

$$= R\left\{ \frac{Sb}{(\alpha_2-\alpha_1)} \left[\frac{i\alpha_1(a+\alpha_1 b)}{z_1+W_1} - \frac{i\alpha_2(a+\alpha_2 b)}{z_2+W_2} \right] \right\} + Sy \ . \qquad (123)$$

A comparison of equations (122) and (123) with the results of Smith[13] now shows them to be the same so that the subsequent differentiation must lead to his stresses for the orthotropic plate with an elliptic hole.

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[13] Ibid., p. 6.

Circular Hole in Orthotropic Plate

If the hole is circular instead of elliptic, equations (120) are modified by setting $b = a$ so that

$$A_1' = \frac{Sa(1+\alpha_1)}{2(\alpha_2^2-\alpha_1^2)} \quad , \qquad B_1' = \frac{-Sa(1+\alpha_2)}{2(\alpha_2^2-\alpha_1^2)}$$

$$A_{-1}' = \frac{Sa}{2(\alpha_2^2-\alpha_1^2)}\left[1+\alpha_1+2\alpha_2\right]$$

$$B_{-1}' = \frac{-Sa}{2(\alpha_2^2-\alpha_1^2)}\left[1+\alpha_2+2\alpha_1\right] .$$

Then equations (98) for the orthotropic plate with a circular hole can be written as

$$\sigma_x = R\left\{-\alpha_1^2\left[\frac{z_1(z_1+W_1')^{A_1'/\lambda_1} - \lambda_1 A_{-1}'}{W_1'(z_1+W_1')}\right]\right.$$

$$\left. - \alpha_2^2\left[\frac{z_2(z_2+W_2')^{B_1'/\lambda_2} - \lambda_2 B_1'}{W_2'(z_2+W_2')}\right]\right\} + \frac{S}{2}$$

$$\sigma_y = R\left\{\left[\frac{z_1(z_1+W_1')^{A_1'/\lambda_1} - \lambda_1 A_1'}{W_1'(z_1+W_1')}\right] + \left[\frac{z_2(z_2+W_2')^{B_1'/\lambda_2} - \lambda_2 B_{-1}'}{W_2'(z_2+W_2')}\right]\right\}$$

$$\tau_{xy} = \frac{R}{2}\left\{-i\alpha_1\left[\frac{z_1(z_1+W_1')^{A_1'/\lambda_1} - \lambda_1 A_{-1}'}{W_1'(z_1+W_1')}\right]\right.$$

$$\left. - i\alpha_2\left[\frac{z_2(z_2+W_2')^{B_1'/\lambda_2} - \lambda_2 B_1'}{W_2'(z_2+W_2')}\right]\right\}$$

where $\lambda_1 = a(1+\alpha_1)$, $\lambda_2 = a(1+\alpha_2)$

and W_1' and W_2' are given by (107).

Further substitutions for A_1', B_1', A_{-1}' and B_{-1}' yield the expressions

$$\sigma_x = R\left\{\frac{-S\alpha_1^2}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{z_1(z_1+W_1')-a^2(1+\alpha_1)(1+\alpha_1+2\alpha_2)}{W_1'(z_1+W_1')}\right]\right.$$

$$\left.+\frac{S\alpha_2^2}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{z_2(z_2+W_2')-a^2(1+\alpha_2)(1+\alpha_2+2\alpha_1)}{W_2'(z_2+W_2')}\right]\right\}+\frac{S}{2}$$

$$\sigma_y = R\left\{\frac{S}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{z_1(z_1+W_1')-a^2(1+\alpha_1)(1+\alpha_1+2\alpha_2)}{W_1'(z_1+W_1')}\right.\right.$$

$$\left.\left.-\frac{z_2(z_2+W_2')-a^2(1+\alpha_2)(1+\alpha_2+2\alpha_1)}{W_2'(z_2+W_2')}\right]\right\} \tag{124}$$

$$\tau_{xy} = \frac{R}{2}\left\{\frac{-i\alpha_1 S}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{z_1(z_1+W_1')-a^2(1+\alpha_1)(1+\alpha_1+2\alpha_2)}{W_1'(z_1+W_1')}\right]\right.$$

$$\left.+\frac{i\alpha_2 S}{2(\alpha_2^2-\alpha_1^2)}\left[\frac{z_2(z_2+W_2')-a^2(1+\alpha_2)(1+\alpha_2+2\alpha_1)}{W_2'(z_2+W_2')}\right]\right\}$$

As was seen for the elliptical hole, more convincing results for the circular hole can be obtained by a development from $\frac{\partial\Phi}{\partial x}$ and $\frac{\partial\Phi}{\partial y}$, i.e.,

$$\frac{\partial\Phi}{\partial x} = R\left\{A_1'\zeta_1+\frac{A_1'}{\zeta_1}+B_1'\zeta_2+\frac{B_1'}{\zeta_2}\right\}$$

$$= R\left\{\frac{Sa}{(\alpha_2-\alpha_1)}\left[\frac{1}{\zeta_1}-\frac{1}{\zeta_2}\right]\right\}$$

which reduces to

$$\frac{\partial \Phi}{\partial x} = R\left\{\frac{S a^2}{(\alpha_2 - \alpha_1)}\left[\frac{(1+\alpha_1)}{z_1 + W_1'} - \frac{(1+\alpha_2)}{z_2 + W_2'}\right]\right\} . \tag{125}$$

Similarly,

$$\frac{\partial \Phi}{\partial y} = R\left\{\frac{S a^2}{(\alpha_2 - \alpha_1)}\left[\frac{i\alpha_1(1+\alpha_1)}{z_1 + W_1'} - \frac{i\alpha_2(1+\alpha_2)}{z_2 + W_2'}\right]\right\} + S y . \tag{126}$$

Appropriate differentiation of (125) and (126) then lead to the same stresses as derived by Professor Smith[14].

As a further check, consider the stresses at the point $z_1 = i\alpha_1 a$, $z_2 = i\alpha_2 a$ in which case

$$W_1' = ia \quad , \quad W_2' = ia$$

$$z_1 + W_1' = ia(1 + \alpha_1)$$

$$z_2 + W_2' = ia(1 + \alpha_2) .$$

Substitution of these values into (124) lead to the expressions

$$\sigma_x = S\,(1 + \alpha_1 + \alpha_2)$$

$$\sigma_y = 0$$

$$\tau_{xy} = 0 .$$

---

[14] *Ibid.*, p. 9.

If the plate itself is isotropic such that

$$\alpha_1 = \alpha_2 = 1$$

then we have the well known results[15]

$$\begin{aligned} \sigma_x &= 3S \\ \sigma_y &= 0 \\ \tau_{xy} &= 0 . \end{aligned}$$

Similarly, for the point $z_1 = a$, $z_2 = a$ it follows that

$$W_1' = \alpha_1 a \quad , \quad W_2' = \alpha_2 a$$

$$z_1 + W_1' = a(1 + \alpha_1)$$

$$z_2 + W_2' = a(1 + \alpha_2)$$

and the stresses (124) become

$$\begin{aligned} \sigma_x &= 0 \\ \sigma_y &= \frac{-S}{\alpha_1 \alpha_2} \\ \tau_{xy} &= 0 . \end{aligned}$$

For the isotropic plate these reduce to

$$\begin{aligned} \sigma_x &= 0 \\ \sigma_y &= -S \\ \tau_{xy} &= 0 \quad . \end{aligned}$$

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[15] For example see: C. T. Wang, Applied Elasticity, New York: McGraw Hill Company, 1953, p. 62.

Summary

In brief summarization, this dissertation provides expressions for stresses in the following elasticity problems:

1. An elliptic disk of orthotropic material in an orthotropic plate.
2. A circular disk of orthotropic material in an orthotropic plate.
3. An elliptic disk of isotropic material in an orthotropic plate.
4. A circular disk of isotropic material in an orthotropic plate.
5. An elliptic hole in an orthotropic plate.
6. A circular hole in an orthotropic plate.

For the reader who is inclined to investigate further, four other cases could be derived, viz.,

7. A rigid elliptic disk in an orthotropic plate.
8. A rigid circular disk in an orthotropic plate.
9. A rigid elliptic disk in an isotropic plate.
10. A rigid circular disk in an isotropic plate.

In the preceding cases of the rigid disks it is important to note that Young's moduli for the disk materials approach ∞ which would impose other obvious modifications on the disk constants.

As further exercises, the reader may wish to reduce items 1 - 4 to the situations in which the plate becomes isotropic.

APPENDIX

General Notation

a , b	lengths of semi-major and semi-minor axes of ellipse as shown in figure 1.
A_n , B_n	coefficients in the series denoting stress for the plate.
$\overset{\circ}{A}_n$, $\overset{\circ}{B}_n$	coefficients in the series denoting the stress function for the elliptic disk.
E_x , E_y	Young's moduli at a given point in the plate associated with tension or compression parallel to the x- and y-axes.
$\overset{\circ}{E}_x$, $\overset{\circ}{E}_y$	Young's moduli at a given point in the disk associated with tension or compression parallel to the x- and y-axes.
u , v	displacements of a point in the plate parallel to the x- and y-axes respectively.
$\overset{\circ}{u}$, $\overset{\circ}{v}$	displacements of a point in the disk parallel to the x- and y-axes respectively.
ϵ_x, ϵ_y	components of strain in plate parallel to x- and y-axes.
$\overset{\circ}{\epsilon}_x$, $\overset{\circ}{\epsilon}_y$	components of strain in disk parallel to x- and y-axes.
γ_{xy}	shearing strain for plate in XY-plane.
$\overset{\circ}{\gamma}_{xy}$	shearing strain for disk in XY-plane.
μ_{xy}	modulus of rigidity associated with the shearing strain in the XY-plane of the plate.
$\overset{\circ}{\mu}_{xy}$	modulus of rigidity in the disk.

σ_x, σ_y	normal components of stress parallel to x- and y-axes in plate.
$\overset{\circ}{\sigma}_x, \overset{\circ}{\sigma}_y$	normal components of stress parallel to x- and y-axes in disk.
τ_{xy}	shearing stress component for plate in the XY-plane.
$\overset{\circ}{\tau}_{xy}$	shearing stress component in the XY-plane of disk.
ν_{xy}	the Poisson's ratio in the plate given by the ratio of the contraction parallel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-axis.
$\overset{\circ}{\nu}_{xy}$	the Poisson's ratio in the elliptic disk.

$$K = \frac{1}{2\sqrt{s_{11} s_{22}}} \left(2 s_{12} + s_{33}\right) \quad , \quad s_{22} = \frac{1}{E_y}$$

$$s_{11} = \frac{1}{E_x} \quad , \quad s_{12} = \frac{-\nu_{xy}}{E_x} = \frac{-\nu_{yx}}{E_y} \quad , \quad s_{33} = \frac{1}{\mu_{xy}}$$

$$\alpha = \sqrt{K + \sqrt{K^2 - 1}} \quad , \quad \beta = \sqrt{K - \sqrt{K^2 - 1}}$$

$$\alpha_1 = \alpha \epsilon \quad , \quad \alpha_2 = \beta \epsilon$$

$$\epsilon = \left(\frac{s_{22}}{s_{11}}\right)^{1/4} = \left(\frac{E_x}{E_y}\right)^{1/4}$$

$$\mathring{K} = \frac{1}{2\sqrt{\mathring{s}_{11}\,\mathring{s}_{22}}}\left(2\mathring{s}_{12} + \mathring{s}_{33}\right) \quad , \qquad \mathring{s}_{33} = \frac{1}{\mathring{\mu}_{xy}}$$

$$\mathring{s}_{11} = \frac{1}{\mathring{E}_x} \quad , \quad \mathring{s}_{12} = \frac{-\mathring{\nu}_{xy}}{\mathring{E}_x} = \frac{-\mathring{\nu}_{yx}}{\mathring{E}_y} \quad , \quad \mathring{s}_{22} = \frac{1}{\mathring{E}_y}$$

$$\mathring{\alpha} = \sqrt{\mathring{K} + \sqrt{\mathring{K}^2 - 1}} \quad , \qquad \mathring{\beta} = \sqrt{\mathring{K} - \sqrt{\mathring{K}^2 - 1}}$$

$$\mathring{\alpha}_1 = \mathring{\alpha}\,\mathring{\epsilon} \quad , \qquad \mathring{\alpha}_2 = \mathring{\beta}\,\mathring{\epsilon}$$

$$\mathring{\epsilon} = \left(\frac{\mathring{s}_{22}}{\mathring{s}_{11}}\right)^{1/4} = \left(\frac{\mathring{E}_x}{\mathring{E}_y}\right)^{1/4}$$

$$\lambda_1 = a + \alpha_1 b \quad , \qquad \mu_1 = a - \alpha_1 b$$

$$\lambda_2 = a + \alpha_2 b \quad , \qquad \mu_2 = a - \alpha_2 b$$

$$\mathring{\lambda}_1 = a + \mathring{\alpha}_1 b \quad , \qquad \mathring{\mu}_1 = a - \mathring{\alpha}_1 b$$

$$\mathring{\lambda}_2 = a + \mathring{\alpha}_2 b \quad , \qquad \mathring{\mu}_2 = a - \mathring{\alpha}_2 b$$

$$\zeta_1 = \frac{z_1 + \sqrt{z_1^2 - \lambda_1\mu_1}}{\lambda_1} \quad , \qquad \zeta_2 = \frac{z_2 + \sqrt{z_2^2 - \lambda_2\mu_2}}{\lambda_2}$$

$$\mathring{\zeta}_1 = \frac{\mathring{z}_1 + \sqrt{\mathring{z}_1^2 - \mathring{\lambda}_1\mathring{\mu}_1}}{\mathring{\lambda}_1} \quad , \qquad \mathring{\zeta}_2 = \frac{\mathring{z}_2 + \sqrt{\mathring{z}_2^2 - \mathring{\lambda}_2\mathring{\mu}_2}}{\mathring{\lambda}_2}$$

In general, a naught superscript on (or above) a symbol denotes relationship to the elliptic disk within the plate.

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BIOGRAPHICAL SKETCH

Leonidas Howard Roberts was born to Warren H. and Celeste G. Roberts of Garard's Fort, Greene County, Pennsylvania on February 27, 1921. He was educated in the public schools of Greene County, Pennsylvania and attended Waynesburg College two years prior to entering the Navy in August 1941. After discharge from the Navy in May 1947 he reentered Waynesburg College and received a Bachelor of Science (mathematics) degree in 1948. Continuing studies at West Virginia University, he obtained a Master of Science degree in August 1949 and since that time has maintained a teaching position in the Physical Science department at the University of Florida.

The writer has been flying airplanes since 1938 and served as a Naval Aviator during his nearly six years active duty. Military assignments included service as a flight instructor at Pensacola, Florida and Banana River (Cocoa), Florida. In 1944-45 he spent fourteen months throughout the Pacific Theatre (to the coast of China) as a Patrol Plane Commander of PBM (Mariner type) multi-engine seaplanes during which time he earned the Distinguished Flying Cross, Air Medal, and a Letter of Commendation from the Commander of the United States 7th Fleet. Promotion to the rank of Lieutenant Commander was attained in June 1946. The writer continues his aviation participation in private life as a CAA designated Commercial pilot with Instrument and Flight Instructor ratings.

In August 1945 he was united in marriage to Mary Brownlee Fabisinski of Pensacola, Florida. Two children, David Howard (age 4) and Claudia Leslie (age 2) complete the family.

The writer is a member of the American Association of University Professors, the Florida Academy of Sciences, and the American Astronomical Society.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 9, 1958

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